**Professor Doctor Habilitate Dumitru Botnaru**
**at his 80th anniversary**

Professor Dumitru Botnaru is one of the Moldavian members of ROMAI. By this, we mean that he lives in Republic of Moldova, more precise in its Capital, Chișinău. We remind here that Romanian Society of Applied and Industrial Mathematics -ROMAI is a Society that has two wings – one in Romania, and the other in Republic of Moldova. From its founding, in 1993, it included members from both sister countries. Moreover, Moldavian members involved themselves in the activity of ROMAI in a vivid and benefic way, by organizing many editions of the annual conference of the Society (Conference of Applied and Industrial Mathematics - CAIM), in Republic of Moldova, by participating in the editions of CAIM in Romania, by publishing in ROMAI Journal, and so on.

Being among the first Moldavian members of ROMAI, Professor Botnaru became a friend of our Society. A reciprocal sympathy relation connected Professor Dumitru Botnaru (and his wife) to the Professor Adelina Georgescu, the founder of ROMAI, that left this world in 2010, and whom we all miss. Meeting each other at CAIM conferences either in Romania or in Republic of Moldova, they always discussed, besides Mathematics, problems concerning the Romanian people from both sides of river Prut. Since ROMAI was founded in 1993, that is short time after the fall of the communism, everyone in the two countries was, at that time, animated by vivid hopes concerning the future. It was a period of grace in which, us, Romanians, tried to conceive a beautiful future of our sister countries.

We present here a few facts concerning Professor's Dumitru Botnaru life and activity. He was born on October 13, 1946, in Dondușeni (in the North of the actual Republic of Moldova) as the youngest of 9 siblings. From his youth he loved both Mathematics and Literature. He was reading classic literature, poetry and novels as well. But the passion for Mathematics was stronger, and, in 1965, he became student of the Faculty of Physics and Mathematics from the State Pedagogical University "Alecu Russo" of Bălţi. Here he was designated as the best student of the Faculty and was sent, in 1968, as a sign of recognition of his value, to continue his studies în Moscow, in the renowned State Pedagogical University.

After graduation, between 1971 and 1973, he was Assistant Lecturer at the Pedagogical University of Bălţi. Then he returned to Moscow, to the same Pedagogical Institute, to pursue doctoral studies (1973-1976). His Doctoral Supervisor was Professor Dmitrii Abrumovici Raicov, a known specialist în Functional Analysis, and the title of his PhD Thesis was *"Bicategorical Structures and Pairs of Adjoint Subcategories*".

With his PhD Degree obtained in 1979 (a confirmation of his mathematical talent and determination to work), Professor Dumitru Botnaru carried out his activity at the Technical University of Moldova, from Chișinău, firstly as a Assistant Lecturer (until 1981) then as a Senior Lecturer (1981-1982), then as an Associated Professor (1982-2002) and, finally as Professor (2002-2016).

In 1992 he became a *Doctoral Supervisor*, and had two PhD Students, that are now Doctors in Mathematics Mrs. Olga Cerbu, Associate Professor at Moldova State University, and Mrs. Alina Țurcanu, Associate Professor at Technical University of Moldova.

In 2001 he was awarded, by the *"Vladimir Andruchievici" Institute of Mathematics and Computer Science, of the Academy of Sciences of Republic of Moldova*, the title of *Doctor Habilitate in the Sciences of Physics and Mathematics*, as a high recognition of his Scientific and Pedagogic activity. His Habilitation Thesis was titled: *"Local, Spectral, and Nuclear Dualities",* and was elaborated under the guidance of Academician Mitrofan Choban (that was, for a long period, Vice President of ROMAI).

Besides the Technical University of Moldova, Prof. Dr. Hab. Dumitru Botnaru, also held courses at the Academy of Transport, Informatics and Communications, and at the former State University of Tiraspol, displaced in Chișinău.

A brief history of his activity in Category Theory is presented in his own paper "*Factorization structures. History and Results"* contained in this issue of ROMAI Journal, pages 1-9.

His very fruitful mathematical activity resulted in a large number of scientific papers published in Mathematical Journals, from which we cite below only some.


1. *On generalized bicategory structures* / Dumitru Botnaru // Soviet Mathematics. Doklady Akademii Nauk SSSR. – 1974. – Vol. 15, nr. 2. – P. 1562-1564.

2. *Reflective subcategories and right bicategory structures* / Dumitru Botnaru // Soviet Mathematics. Doklady Akademii Nauk SSSR. – 1974. – Vol. 15, nr 6. – P. 1588-1596.

3. *Quasi-identity and identity in the category of dynamics* / Dumitru Botnaru // Questions and Answers in General Topology. – 1990. – Vol. 8, no 1. – P. 19-32. – ISSN 0918-4732.

4. *Categoria spațiilor A-vagi* / Dumitru Botnaru // Buletin științific. Seria : Matematică și Informatică / Universitatea din Pitești. – 1998. – Nr 2. – P. 23-36. – ISSN 1453-116x.

**5.** *Dualité locale et nucléaire* / Dumitru Botnaru // Buletin științific. Seria : Matematică și Informatică / Universitatea din Pitești. – 2000. – Nr 6. – P. 37-58. – ISSN 1453-116x.

6. *Dualité locale* / Dumitru Botnaru // Scientific Annals. Faculty of Mathematics and Informatics, Moldova State University. – 2000. – Vol. 2, nr 1. – P. 66-84.

7. *Equivalence des triplets dans la dualite locale* / Dumitru Botnaru // Analele științifice ale USM. Seria : Științe fizico-matematice. – Chișinău, 2001. – P. 19-23. – ISBN 9975-70-042-x.

8. *Dualité spectrale. Le cas covariant* / Dumitru Botnaru // Scripta Scientiarum Mathematicarum : Scripta seminarii investigationum scientiae. Topologiae Systematum Subtilium et Applicationum. – Chişinău : Tehnica-Info, 2001. – Tomus 2, Fasciculus 1. – P. 199-221. – ISBN 9975-63-073-1.

9. *The R - Perfect Morphismes* / Dumitru Botnaru, Alina Țurcanu // Acta Universitatis Apulensis: Mathematics - Informatics / Universitatea „1 Decembrie 1918" din Alba Iulia. – 2004. – No 7. – P. 49-53. – ISSN 1582-5329.

10. *On Giraux subcategories in locally convex spaces* / Dumitru Botnaru, Alina Țurcanu // ROMAI Journal. – 2005. – Vol. 1, no 1. – P. 7-30. – ISSN 1841-5512.

11. *Bicategory structures generated by injective spaces* / Dumitru Botnaru, Alina Țurcanu, Olga Cerbu // ROMAI Journal. – 2007. – Vol. 3, nr 2. – P. 31-50. – ISSN 1841-5512.

12. *Structures bicatégorielles complémentaires* / Dumitru Botnaru // ROMAI Journal. – 2009. – Vol. 5, nr 2. – P. 5-27. – ISSN 1841-5512.

13. *Semireflexive subcategories* / Dumitru Botnaru, Olga Cerbu // ROMAI Journal. – 2009. – Vol. 5, nr 1. – P. 7-20. – ISSN 1841-5512.

14. *The factorization of the right product of two subcategories* / Dumitru Botnaru, Alina Țurcanu // ROMAI Journal. – 2010. – Vol. 6, nr 2. – P. 41-53. – ISSN 1841-5512.

15. *Structures bicatégorielles complémentaires* / Dumitru Botnaru // Revue Roumaine Mathématiques Pures et Appliquées. – 2010. – Tome LV, no 2. – P. 97-119. – ISSN 0035-3965.

16. *The Hewitt Spaces as Right Factorized Product of Two Subcategories* / Dumitru Botnaru, Alina Țurcanu // Buletinul Institutului Politehnic din Iași. Secţia: Matematică. Mecanică teoretică. Fizică. – 2011. – Tomul LVII(LXI), Fasc. 1. – P. 37-46. – ISSN 1224-7863.

17. *The Classes of Hereditary Morphisms and the Theory of Relative Torsion* / Dumitru Botnaru, Elena Baeș // Buletinul Institutului Politehnic din Iaşi. Secţia: Matematică. Mecanică teoretică. Fizică. – 2011. – Tomul LVII(LXI), Fasc. 1. – P. 15-28. – ISSN 1224-7863.

18. *Semireflexives Subcategories and the Right Product* / Dumitru Botnaru, Olga Cerbu // Buletinul Institutului Politehnic din Iași. Secţia: Matematică. Mecanică teoretică. Fizică. – 2011. – Tomul LVII(LXI), Fasc. 1. – P. 29-35. – ISSN 1224-7863.

19. *Categorial aspects of the semireflexivity* / Dumitru Botnaru, Olga Cerbu // Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica. – 2011. – Nr 3(67). – P. 77-83. – ISSN 1024-7696.

20. *Factorization structures with nonhereditary classe of projections* / Dumitru Botnaru, Elena Baeş // ROMAI Journal. – 2012. – Vol. 8, nr 2. – P. 27-37. – ISSN 1841-5512.

21. *On the right product of two subcategories* / Dumitru Botnaru, Alina Țurcanu // ROMAI Journal. – 2016. – Vol. 12, no 2. – P. 9-18. – ISSN 1841-5512.

22. *Souscatégories $\mathcal{L}$-semi-reflexives* / Dumitru Botnaru // Acta et Commentationes: Ştiinţe Exacte şi ale Naturii. – 2017. – Nr 2(4). – P. 11-34. – ISSN 2537-6284.

23. *Some properties of left product of two subcategories* / Dumitru Botnaru, Alina Țurcanu // Acta et Commentationes: Ştiinţe Exacte şi ale Naturii. – 2017. – Nr 2(4). – P. 35-44. – ISSN 2537-6284.

24. *The center of the lattice of factorization structures* / Dumitru Botnaru, Elena Baeş // Buletinul Academiei se Ştiinţe a Republicii Moldova. Matematica. – 2018. – Nr 1(86). – P. 50-66. – ISSN 1024-7696.

25. *The cartesian product of two subcategories = Produsul cartezian a două subcategorii* / Dumitru Botnaru, Olga Cerbu // Acta et commentationes: Ştiinţe Exacte și ale Naturii. – 2018. – Nr 2(6). – P. 133-138. – ISSN 2537-6284.

26. *Noyaux des sous-catégories semi-réflexives* / Dumitru Botnaru // ROMAI Journal. – 2018. – Vol. 14, no 2. – P. 1-32. – ISSN 1841-5512.

**27.** *Groupoïd des sous-catégories $\mathcal{L}$-semi-reflexives* (*The Groupoïd of $\mathcal{L}$-semireflexive subcategories*) / Dumitru Botnaru // Revue Roumaine de

Mathematiques Pures et Appliquees. – 2018. – Nr 1(63). – P. 61-71. – ISSN 0035-3965.

28. *Couples des sous-catégories conjuguées* / Dumitru Botnaru // ROMAI Journal. – 2018. – Vol. 14, no 1. – P. 23-41. – ISSN 1841-5512.

29. *La classe des sous-catégories $\mathscr{L}$-semi-réflexives* / Dumitru Botnaru // ROMAI Journal. – 2020. – Vol. 16, no 1. – P. 19-36. – ISSN 1841-5512.

30. *Couples des souscatégories bisemiréflexives et les théories de torsion relative* / Dumitru Botnaru // ROMAI Journal. – 2021. – Vol. 17, no 1. – P. 29-39. – ISSN 1841-5512.

31. *Variétés $\mathscr{L}$-semi-réflexives* / Dumitru Botnaru // ROMAI Journal. – 2022. – Vol. 18, no 2. – P. 57-71. – ISSN 1841-5512.

32. *Commutative monoids of reflective functors* / Dumitru Botnaru, Alina Turcanu // ROMAI Journal. – 2022. – Vol. 18, no 2. – P. 73-80. – ISSN 1841-5512.

33. *Certaines modalités pour définir les sous-catégories semi-réflexives* / Dumitru Botnaru // ROMAI Journal. – 2022. – Vol. 18, no 1. – P. 9-15. – ISSN 1841-5512.

34. *On semireflexive subcategories* / Dumitru Botnaru // Topology and its Applications. – 2023. – Vol. 340 (1 December). – Article 108713. – ISSN 0166-8641. – E-ISSN1879-3207. DOI: https://doi.org/10.1016/j.topol.2023.108713. – URL: https://www.sciencedirect.com/science/article/abs/pii/S0166864123003061.

35. *La catégorie des espaces $\mathscr{B}$-inductifs semi-reflexifs* / Dumitru Botnaru // ROMAI Jounal. – 2023. – Vol. 19, no 1. – P. 63-71. – ISSN 1841-5512.

In 2023, Professor Botnaru published a very comprehensive monograph on Category Theory, written in Romanian:

*Aspecte categoriale ale spațiilor topologice vectoriale : Subcategorii semireflexive = Les aspectes catégoriels des espaces topologiques vectoriels : Sous-catégories semi-reflexives* / Dumitru Botnaru. – Chişinău : Ştiinţa, 2023. – 484 p. – Bibliogr.: p. 467-472. – ISBN 978-9975-85-373-6 (https://www.romai.ro/src/library/Botnaru_monografie.pdf).

Besides these scientific publications we must remind the great number of communications at Conferences, and a number of 10 books, (and some articles) for the undergraduate level (high-school and gymnasium).

Professor Dr. Habilitate Dumitru Botnaru is, now, at the beautiful age of 80.

In the name of all ROMAI members, we wish him to live many beautiful years, in good health, and to enjoy the fruits of his work!

*Anca-Veronica Ion,* Editor of ROMAI Journal

*Alina Țurcanu*, Associate Professor, Technical University of Moldova

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# FACTORIZATION STRUCTURES. HISTORY AND RESULTS

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**Abstract** **Editors' note:** This work is a review of Professor's Dumitru Botnaru Mathematical activity and it is an Invited Paper in our Journal, with the occasion of his $80^{th}$ anniversary.

**Keywords:** the orthogonal morphisms, the reflective, coreflective, semireflexive, c-reflective subcategories, the semireflexive products of two subcategories, the factorization structures, the Hausdorff locally convex topological vector spaces.
**2020 MSC:** 46 M 15, 18 F 60.

## 1. INTRODUCTION

In 1967, I was a student at the University of Bălţi and, within the framework of the national staff training program, I was transferred to the Moscow Pedagogical University. In Bălţi, my teacher was the Associate Professor Grigorie Haimonovici Berman who had defended his thesis at the Moscow Pedagogical University under Professor Dmitrii Abrumovici Raicov, a great specialist in Functional Analysis. The thesis consisted in the construction and study of the properties of some functors in the category of Banach spaces. He directed me to attend the seminars of Professor D. Raicov, that held a theoretical course and a seminar at which doctoral candidates and students presented papers.

Among the professors in Bălţi I would like to mention Professor Melihson, a person with studies in France and who could orally cube a three-digit number and do it faster than us students in the benches with a pen. In 1967, the head of the Algebra Department of the Lomonosov University, A. Curoș had published in Uspehi Mat. Nauk. an extensive article on algebraic categories, the first publication in Russian in the field of Category Theory.

Professor Raicov D. asked me if I knew Romanian. He was from Odessa (born in 1910). So, he knew very well that Bessarabia was Romanian land. The Faculty of Mathematics subscribed to both Romanian journals: *Studii şi Cercetări Matematice* and *Revue Roumaine de Mathématiques Pures et Apliquées.*

In the monograph of Prof. Cabiria Cazacu, when examining additive categories, the problem of connection morphism $\alpha$ appears for the first time.

Let $\alpha : X \to Y$ be a morphism in an additive category with kernels and cokernels. Then we have the following diagram.

$$\begin{tikzcd}
Ker\,\alpha \arrow[r, "ker\alpha"] & X \arrow[r, "\alpha"] \arrow[d, "cok\,ker\alpha"'] & Y \arrow[r, "cok\,\alpha"] & Cok\,\alpha \\
 & Cok\,ker\alpha \arrow[r, "\overline{\alpha}"] & Ker\,cok\,\alpha \arrow[u, "ker\,cok\,\alpha"'] &
\end{tikzcd}$$

Romanian scientists Jurchescu M. and Lascu A., [4] examined the case when $\bar{\alpha}$ is a bimorphism.

Professor D. Raicov proposed the following situation, which bears the name of semiabelian category in the sense of Raicov [6]:

- $\mathcal{C}$ is a category with null object, finite projective and inductive limits;

- the class of kernels $\mathcal{E}q(\mathcal{C})$ is stable on the right, that is, in any pushout square $\alpha \cdot \beta' = \beta \cdot \alpha'$, if $\alpha \in \mathcal{E}q(\mathcal{C})$, then also $\alpha' \in \mathcal{E}_q(\mathcal{C})$, and the class $\mathcal{C}oeq(\mathcal{C})$ is stable on the left;

- for any morphism $\alpha \in \mathcal{C}$ the morphismis $\bar{\alpha}$ is a bimorphism. There was also an axiom, which allowed to prove that $Hom(X, Y)$ is an abelian group.

A disciple of Professor Raicov proved that the category, let us note it $\mathcal{C}_2\mathcal{V}$ of locally convex topological vector spaces Hausdorff, is a semiabelian category in the Raicov sense. Thus the study of this category began.

The two Romanian scientists mentioned above also introduced the notions: homotopic and cohomotopic with zero, an article published, firstly, in Romanian and then in a German journal.

After I presented these notions at the seminar, Professor Raicov proposed to simplify these notions in the following way [7]: A morphism $\alpha$ is considered orthogonal from above to a morphism $\beta$ and $\beta$ orthogonal from below to $\alpha$ if for any commutative square

$$\gamma \cdot \alpha = \beta \cdot \delta$$

there exists its diagonal $\varphi$ such that:

$$\begin{tikzcd}
\cdot \arrow[r, "\alpha"] \arrow[d, "\delta"'] & \cdot \arrow[d, "\gamma"] \arrow[dl, dashed, "\varphi"'] \\
\cdot \arrow[r, "\beta"] & \cdot
\end{tikzcd}$$

$$\delta = \varphi \cdot \alpha, \ \gamma = \beta \cdot \varphi$$

For any class of morphisms $\mathcal{A}$, the notations were introduced:

$\mathcal{A}^{\top}$ - all orthogonal morphisms from above to any morphism in $\mathcal{A}$;

$\mathcal{A}^{\perp}$ - all orthogonal morphisms from below to any morphism in $\mathcal{A}$;

$\mathcal{A}^{\urcorner} = \mathcal{E}pi \cap \mathcal{A}^{\top}$,

$\mathcal{A}^{\llcorner} = \mathcal{M}ono \cap \mathcal{A}^{\perp}$.

I have observed that for any factorization structure $(\mathcal{P}, \mathcal{I})$ we have $\mathcal{P} = \mathcal{I}^{\urcorner}$ and $\mathcal{I} = \mathcal{P}^{\llcorner}$.

For right factorization structures $(\mathcal{P}, \mathcal{I})$ we have $\mathcal{P} = \mathcal{I}^{\urcorner}$ and $\mathcal{I} = \mathcal{P}^{\perp}$.

For left factorization structures $(\mathcal{P}, \mathcal{I})$ we have $\mathcal{P} = \mathcal{I}^{\top}$ and $\mathcal{I} = \mathcal{P}^{\llcorner}$.

I have formulated and demonstrated the following

**Theorem.** *In a small locally and colocallly category with projective (respectively inductive) limits for any class of epi $\mathcal{A}$ (respectively for any class of mono $\mathcal{A}$) the pair $(\mathcal{A}^{\perp\urcorner}, \mathcal{A}^{\perp})$, respectively $(\mathcal{A}^{\urcorner}, \mathcal{A}^{\urcorner\perp})$, is a left (respectively right) factorization structure. In particular, in a small locally and colocal category with projective or inductive limits for any class of epi $\mathcal{E}$ and any class of mono $\mathcal{M}$ the pairs $(\mathcal{E}^{\perp\urcorner}, \mathcal{E}^{\llcorner})$, and $(\mathcal{M}^{\urcorner}, \mathcal{M}^{\urcorner\llcorner})$ form factorization structures.*

Professor Raicov published this statement calling it the Kelly-Botnaru Theorem. Mathematician Kelly proved this statement in the case when $\mathcal{E} = \mathcal{E}pi$ and $\mathcal{M} = \mathcal{M}ono$. Thus we began the study of factorization structures (initially called bicategorial structures) in the category $\mathcal{C}_2\mathcal{V}$.

## 2. FACTORIZATION STRUCTURES IN THE CATEGORY $\mathcal{C}_2\mathcal{V}$

The notion of factorization structure is closely related to that of reflective subcategory.

**Definition.** *Let $\mathcal{A}$ be a subcategory of the category $\mathcal{C}$ with the inclusion functor $i : \mathcal{A} \to \mathcal{C}$, which admits a left adjoint $\mathcal{F} : \mathcal{C} \to \mathcal{A}$.*

*- $\mathcal{F}(X)$ is called a free object over the object $X$ with the adjunction morphism $w^X : X \to \mathcal{F}(X)$.*

*- If $\mathcal{A}$ it is a full subcategory of the category $\mathcal{C}$, then it is called a reflective subcategory and $w^X : X \to \mathcal{F}(X)$ is called a replica of the object $X$.*

**Definition.** *Let $(\mathcal{P}, \mathcal{I})$ be a right factorization structure and $\mathcal{R}$ be a reflective subcategory of the category $\mathcal{C}$. $\mathcal{R}$ is called $\mathcal{P}$-reflective if for any object $X \in \mathcal{C}$ $\mathcal{R}$-replica $w^X : X \to \mathcal{F}(X)$ belongs to the class $\mathcal{P}$.*

**Theorem** (Kennison). *Let $\mathcal{C}$ be a category with products, $(\mathcal{P}, \mathcal{I})$ a right factorization structure. $\mathcal{R}$ is a $\mathcal{P}$-reflective subcategory if and only if it is closed under products and $\mathcal{I}$-subojects.*

It was quickly established that in the category $\mathcal{C}_2\mathcal{V}$ there is a proper class (not a set) of reflective subcategories.

The hypothesis was formulated: since the pair $(\mathcal{E}_u, \mathcal{M}_p) =$ (surjective mappings, topological inclusions) is a factorization structure, the projection class $\mathcal{E}_u$ is left stable, if the pair of dual classes $(\mathcal{E}_p, \mathcal{M}_u)$ is also a factorization structure.Together with a colleague of mine, also a student, I described these classes. The results were published in the Bulletin of the Institute of Mathematics of the Academy of Sciences of the Republic of Moldova.

The monomorphism $m : X \to Y$ belongs to the class $\mathcal{M}_u$, iff, any morphism $v : X \to K$ (K – the field of real numbers) extends through $m$.

- The epimorphism $e : X \to Y$ of the class $\mathcal{E}_u$ belongs to the class $\mathcal{E}_p$ iff the square

$$e \cdot m^X = m^Y \cdot m(e)$$

$$\begin{tikzcd}
mX \arrow[r, "m(e)"] \arrow[d, "m^X"'] & mY \arrow[d, "m^Y"] \\
X \arrow[r, "e"] & Y
\end{tikzcd}$$

is pullback, where $m^X$ and $m^Y$ are the $\widetilde{\mathcal{M}}$-coreplicas of the respective objectives ($\widetilde{\mathcal{M}}$ - the coreflective subcategory of spaces with Mackey topology).

-The pair $(\mathcal{E}_p, \mathcal{M}_u)$ represents a factorization structure in the category $\mathcal{C}_2\mathcal{V}$.

Let $f : X \to Z$ be a morphism of the category $\mathcal{C}_2\mathcal{V}$, $f = i \cdot p$ $(\mathcal{E}_u, \mathcal{M}_p)$-factorization of $f$.

$$\begin{tikzcd}
mX \arrow[rr, "m(p)"] \arrow[dd, "m^X"'] & & mY \arrow[dl, dashed, "v"'] \arrow[dd, "m^Y"] & \\
& {} \arrow[dr, dashed, "t"] & & \\
X \arrow[ur, dashed, "u"] \arrow[rr, "p"] \arrow[rrr, bend right, "f"'] & & Y \arrow[r, "i"] & Z
\end{tikzcd}$$

We examine the $\widetilde{\mathcal{M}}$-coreplicas of objects $X$ and $Y$, then

$$p \cdot m^X = m^Y \cdot m(p).$$

On the morphisms $m^X$ and $m(p)$ construct the pullback square

$$u \cdot m^X = v \cdot m(p).$$

Then

$$p = t \cdot u,$$

$$m^Y = t \cdot v$$

for un $t$. From the last equality it follows that $t \in \mathcal{E}_u \cap \mathcal{M}_u$, and

$$f = (i \cdot t) \cdot u$$

is the $(\mathcal{E}_p, \mathcal{M}_u)$-factorization of $f$.

The theorem below is also proved in an analogous way.

**Theorem.** ([2], Th. 6.6.9). *For any nonzero reflective subcategory $\mathcal{R}$ of the category $\mathcal{C}_2\mathcal{V}$ the class of factorization structures $(\mathcal{P}, \mathcal{I})$, for which $\mathcal{R}$ is $\mathcal{P}$-reflective, possesses the smallest element $(\mathcal{P}'(\mathcal{R}), \mathcal{I}'(\mathcal{R}))$ and the largest element $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$.*

The following theorems describe these factorization structures for both a reflective and a coreflective subcategory.

Let $\mathcal{R}$ be nonzero reflective subcategory,

$$\mathcal{U}(\mathcal{R}) = \{r^X : X \to rX | X \in |\mathcal{C}_2\mathcal{V}|\},$$

$$\mathcal{V}(\mathcal{R}) = \{v^X : rX \to \pi X | X \in |\mathcal{C}_2\mathcal{V}|\},$$

where $\pi : X \to \Pi$ is the reflector functor and $\Pi$ is the subcategory of complete spaces with weak topology. Then

$$\mathcal{I}'(\mathcal{R}) = (\mathcal{U}(\mathcal{R}))^{\llcorner}, \ \mathcal{P}'(\mathcal{R}) = (\mathcal{I}'(\mathcal{R}))^{\urcorner},$$
$$\mathcal{P}''(\mathcal{R}) = (\mathcal{V}(\mathcal{R}))^{\urcorner}, \ \mathcal{I}''(\mathcal{R}) = (\mathcal{P}''(\mathcal{R}))^{\llcorner}.$$

A special role in my research was played by semireflexive subcategories. My monograph also has a subtitle "Semireflexive subcategories".

Let $\mathcal{L}$ be a nonzero reflective subcategory (respectively $\mathcal{K}$ a coreflective subcategory) with the reflector functor $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$, (respectively the coreflective functor $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$. Let us denote

$$\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi | r(e) \in \mathcal{I}so\},$$

$$\mu\mathcal{K} = \{m \in \mathcal{M}ono | k(m) \in \mathcal{I}so\}.$$

The morphism $e : X \to Y \in \varepsilon\mathcal{R}$, iff $e = f \cdot r^X$ for a morphism $f$.

The morphism $m : X \to Y \in \mu\mathcal{K}$, iff $m = k^Y \cdot f$ for a morphism $f$.

**Definition.** *Let $\mathcal{L}$ be a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. The subcategory $\mathcal{R}$ is called $\mathcal{L}$-semireflexive, if it is closed with respect to $\varepsilon\mathcal{L}$-subjects and $\varepsilon\mathcal{L}$-factorobjects.*

*If $\mathcal{L}_1 \subset \mathcal{L}_2$ then $\varepsilon\mathcal{L}_2 \subset \varepsilon\mathcal{L}_1$. Thus, for $\mathcal{L} = \mathcal{C}_2\mathcal{V}$ the $\mathcal{L}$-semireflexive subcategories are all saturated subcategories.*

At first, $\mathcal{S}$-semireflexive subcategories were defined and studied, where $\mathcal{S}$ is the subcategory of spaces with weak topology.

Berezenskii I. A. examined [1] the subcategory $i\mathcal{R}$ of inductively semireflexive spaces and proved that it is an $\mathcal{S}$ -semireflexive subcategory.

At the same time, Brudovskii B. S. [3] proved that the subcategory $\mathcal{S}h$ of Schwartz spaces is also an $\mathcal{S}$ -semireflexive subcategory.

These were the first examples of semireflexive subcategories.

The subcategory $l\Gamma$ of locally complete spaces [10] was studied by D. Raicov [5] from whose results we deduce that this subcategory is $\mathcal{S}$-semireflexive.

V. Sekovanov introduced the notion of $\mathcal{B}$-inductive semireflexive space [8-9], and then I proved that they form an $\mathcal{S}$-semireflexive subcategory.

An important role in my research was played by $c$-reflective subcategories and their class $\mathbb{R}c$.

**Definition.** *Let $\mathcal{L}$ be a reflective subcategory containing the subcategory $\mathcal{S}$ of spaces with weak topology. It is called c-reflective if the reflective functor $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ is exactly left.*

**Theorem.** *Let $\mathcal{L}$ be a reflective subcategory and $\mathcal{S} \subset \mathcal{L}$. Then the reflective functor $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ commutes with products.*

Let $\widetilde{\mathcal{M}}$ be the subcategory of spaces with Mackey topology with the coreflective functor $m : \mathcal{C}_2\mathcal{V} \to \widetilde{\mathcal{M}}$.

**Theorem.** *Let $\mathcal{L}$ be a reflective subcategory and $\mathcal{S} \subset \mathcal{L}$. $\mathcal{L}$ is a c-reflective subcategory iff the class $\varepsilon\mathcal{R}$ is left stable.*

Thus if $\mathcal{L}$ is $c$-reflective, then $(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^{\top})$ is a left factorization structure. Decomposing by this structure the $\widetilde{\mathcal{M}}$ -coreplica of the object $X$,

$$\begin{tikzcd} mX \arrow[r, "t^X"] \arrow[rr, bend right, "m^X"'] & kX \arrow[r, "k^X"] & X \end{tikzcd}$$

we obtain an $(\varepsilon\mathcal{R})$-coreflective subcategory $\mathcal{K}$ and $k^X : kX \to X$ its $\mathcal{K}$-coreplica. The respective functors $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ and $l : \mathcal{C}_2\mathcal{V} \to \mathcal{L}$ possess the properties:

$$l \cdot l \sim k, l \cdot k \sim l$$

Also,

$$(\varepsilon\mathcal{R}) = (\mu\mathcal{K}).$$

Thus we have: The class $\mathcal{R}c$ of $c$-reflective subcategories;

The class $\mathcal{K}c$ of $c$-coreflective subcategories;

The class $\mathcal{P}c$ of $(\mathcal{K}, \mathcal{L})$ pairs of conjugate subcategories, $\mathcal{K} \in \mathbb{K}_c$, $\mathcal{L} \in \mathbb{R}_c$ and $\mu\mathcal{K} = \varepsilon\mathcal{L}$. These are proper classes and are isomorphic to each other.

The semireflective product is the notion that comes to explain the relations between $c$-reflective and semireflective subcategories.

**Definition.** *Let $\mathcal{L}$ and $\mathcal{A}$ be two subcategories and $\mathcal{R} \in \mathbb{R}$. The object $A \in \mathcal{C}_2\mathcal{V}$ is called $(\mathcal{L}, \mathcal{A})$-semireflective if its $\mathcal{L}$-replica belongs to the subcategory $\mathcal{A}$.*

We will denote by $\mathcal{L} *_{sr} \mathcal{A}$ the full subcategory of all $(\mathcal{L}, \mathcal{A})$-semireflective objects. It is called the semireflective product of the subcategories $\mathcal{L}$ and $\mathcal{A}$.

**Theorem.** *Let $\mathcal{L} \in \mathbb{R}_c$ and $\mathcal{R} \in \mathbb{R}$. Then $\mathcal{L} *_{sr} \mathcal{A}$ is a $\mathcal{L}$-semireflective subcategory.*

Thus if $\mathcal{L}$ is a $c$-reflective subcategory, then the semireflective product $\mathcal{L} *_{sr} \mathcal{R}$ applies the class $\mathbb{R}$ to the class $\mathbb{R}_{sr}$ of $\mathcal{L}$-semireflective subcategories.

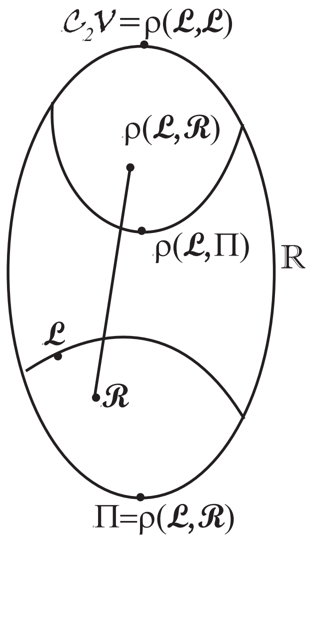
$$\mathbb{R} \xrightarrow{\mathcal{L}^{*}_{sr}} \mathbb{R}_{sr}$$

Some operations on the classes $\mathbb{R}_c$ and $\mathbb{R}$ have been defined and examined.

## 3. THE OPERATION $\rho(\mathcal{L}, \mathcal{R})$

*Notations*: Let $\mathcal{L}$ and $\mathcal{R} \in \mathbb{R}$ be. Then $\rho(\mathcal{L}, \mathcal{R})$ is the full subcategory of all objects $X$ for which $rX = lX$.

**Theorem** ([2],6.4.4). *Let $\mathcal{L} \in \mathbb{R}_c$. Then $\rho(\mathcal{L}, \mathcal{R})$ is an $\mathcal{L}$-semireflexive subcategory.*



## 4. THE COMMUTATIVE SEMIGROUP $\mathbb{V}_{pb}$

*Notations:* Let $\mathbb{B}ic$ be the class of all bicomplete classes of bimorphisms of the category $\mathcal{C}_2\mathcal{V}$, $\mathbb{V}_{pb} = \{\mathcal{S}_{\mathcal{B}}(\mathcal{S})|\mathcal{B} \in \mathbb{B}ic\}$.

**Theorem** ([2], 9.5.12). Consider $\mathcal{V}_1$, $\mathcal{V}_2 \in \mathbb{V}_{pb}$ and $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$·. Then $\mathcal{V} \in \mathbb{V}_{pb}$ and for any object $X \in |\mathcal{C}_2\mathcal{V}|$ we have: $vX = \mathcal{V}$, $v_2X = v_2v_1X$. Thus,

$$v = v_1v_2 = v_2v_1.$$

So, in the class of reflector functors in $\mathbb{V}_{pb}$ the composition operation is defined with the following properties:
- commutative
- associative
- with neutral element $\mathcal{C}_2\mathcal{V} \to \mathcal{C}_2\mathcal{V}$.

It should be noted that out of about 20 disciples of D. Raicov, only I and the Bulgarian Petru Kenderov were not Jews. P. Kenderov presented the theory of locally convex topological vector spaces over the field of real numbers with discrete topology. In the Soviet Union there was a rule - foreign doctoral students for three years had to leave with defended theses. P. Kenderov then became the director of the Institute of Mathematics of the Academy of Sciences in Sofia.

Another example can be the attitude of academician Israil Ţuricovici Gohberg from Chişinău: I am preparing a Moldovan doctor, but, at the same time, a Jew one.

In relation to factorization structures, I would mention the following problem that starts from some notions defined by academician M. Ciobanu in the category of Tihonov spaces, where $\mathcal{T}h$ is the subcategory of Tihonov spaces: subspaces of the space $[0,1]^\tau$ or subspace of the space $\mathbb{R}^\tau$ for a cardinal $\tau$.

**Definition.** 1. *The morphism $f : (X,u) \to (Y,v) \in$ is called $\tau$-fine if for any point $y_0 \in Y \setminus f(x)$ there is a family $\{G_i|i \in |M|\}$, $|M| < \tau$ of open sets in $Y$ such that from the fact that $\cap G_i \neq \varnothing$ it follows that $(\cap G_i) \cap f(x) \neq \varnothing$.*

*We denote $\mathcal{F}(\tau)$ the class of $\tau$-fine morphisms.*

2. *The morphism $f : (X,u) \to (Y,v) \in \mathcal{T}h$ is called $\tau$-closed if for any point $y_0 \in Y \setminus f(x)$ there is a family $\{G_i|i \in |M|\}$ of open sets in $Y$ such that $y_0 \in \cap G_i$ and $(\cap G_i) \cap f(x) \neq \varnothing$.*

We denote $\mathcal{F}'(\tau)$ the class of $\tau$-closed topological inclusions.

Problem. For what cardinals $\tau$ does the pair $(\mathcal{F}(\tau), \mathcal{F}'(\tau))$ form in the category $\mathcal{T}h$ a factorization structure.

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# B-SEMIREFLEXIVE SUBCATEGORIES: CLOSURE THEORY, COMPONENT DECOMPOSITION, AND REFLECTIVE APPLICATIONS

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**Abstract** Let $\mathcal{C}$ be an essentially small category and let $\mathcal{B}$ be a composition-stable class of bimorphisms containing all isomorphisms. A full replete subcategory is called $\mathcal{B}$-semireflexive when it is closed both under $\mathcal{B}$-subobjects and under $\mathcal{B}$-factorobjects. This paper develops a closure-theoretic description of such subcategories and connects it with the theory of reflective and semireflexive constructions. The operators assigning to a subcategory its $\mathcal{B}$-subobjects and its $\mathcal{B}$-factorobjects are shown to be closure operators. The main result characterises $\mathcal{B}$-semireflexive subcategories as unions of connected components of the undirected graph generated by $\mathcal{B}$-morphisms. Consequently, the collection of all $\mathcal{B}$-semireflexive subcategories forms a complete atomic Boolean algebra, and the least $\mathcal{B}$-semireflexive subcategory containing a given class of objects is obtained by finite $\mathcal{B}$-zigzag saturation. The construction is invariant under equivalences of categories and behaves contravariantly with respect to enlargement of the morphism class. For a reflective subcategory $\mathcal{L}$, the theory is applied to the class $\varepsilon_{\mathcal{L}}$ of epimorphisms inverted by the reflector, yielding a component interpretation of $\mathcal{L}$-semireflexivity and of a classical semireflexive-product criterion. Examples from balanced categories and Hausdorff locally convex spaces illustrate both trivial and genuinely nontrivial cases.

**Keywords:** bimorphism; semireflexive subcategory; reflective subcategory; closure operator; factorobject; subobject; categorical equivalence; locally convex space.
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## 1. INTRODUCTION

Reflective subcategories are among the most effective mechanisms for selecting canonical objects inside a category. A reflector replaces an arbitrary object by a universal approximation lying in a prescribed full subcategory, and many familiar constructions—completion, separation, localisation, compactification, or algebraic reflection—fit this pattern. In categories of topological vector spaces, however, the morphisms associated with a reflection frequently admit nontrivial factorizations. This leads naturally to intermediate classes of objects and to closure conditions involving both subobjects and factorobjects.

The theory of semireflexive subcategories developed by Botnaru and Cerbu arose from this interaction between reflections, classes of bimorphisms, and locally convex spaces [3, 4, 5]. If $\mathcal{L}$ is reflective and $\varepsilon_{\mathcal{L}}$ denotes the class of epimorphisms transformed into isomorphisms by the reflector, a subcategory is called $\mathcal{L}$-semireflexive when it is closed with respect to $\varepsilon_{\mathcal{L}}$-subobjects and $\varepsilon_{\mathcal{L}}$-factorobjects. This notion is categorical; it must not be confused with the classical functional-analytic property that a locally convex space maps surjectively onto its bidual. The terminology reflects a common structural idea, but the objects being classified are subcategories rather than individual spaces.

The purpose of the present paper is to isolate the part of the construction that depends only on a class $\mathcal{B}$ of bimorphisms. This abstraction has three advantages. First, it separates the intrinsic closure mechanism from the particular reflector producing the class $\varepsilon_{\mathcal{L}}$. Second, it permits a direct lattice-theoretic treatment. Third, it reveals that $\mathcal{B}$-semireflexivity is controlled by a simple but useful connectivity relation on objects.

The main contribution is the component theorem: a full replete subcategory is $\mathcal{B}$-semireflexive if and only if it is a union of connected components in the undirected graph whose edges are the morphisms belonging to $\mathcal{B}$. This result yields an explicit description of the least semireflexive closure, a complete atomic Boolean algebra of semireflexive subcategories, a finite-zigzag membership criterion, and an efficient procedure in finite categories. It also clarifies why the theory is trivial in balanced categories for the class of all bimorphisms, while it becomes nontrivial in the category of Hausdorff locally convex spaces.

The paper is organised as follows. Section 2 introduces the categorical setting and the two basic closure operators. Section 3 proves the component theorem and its lattice consequences. Section 4 studies changes of the bimorphism class and invariance under equivalence. Section 5 relates the abstract theory to reflective subcategories and semireflexive products. Section 6 gives representative examples, and Section 7 records the finite computational form of the classification.

## 2. CATEGORICAL SETTING AND ELEMENTARY CLOSURE OPERATORS

Throughout the paper, $\mathcal{C}$ is an essentially small category. This assumption avoids set-theoretic difficulties when speaking about the lattice of full replete subcategories. The results remain valid relative to a fixed Grothendieck universe.

A morphism is a *bimorphism* when it is simultaneously a monomorphism and an epimorphism. The category $\mathcal{C}$ is *balanced* if every bimorphism is an isomorphism. We fix a class

$$\mathcal{B} \subseteq \mathrm{Bim}(\mathcal{C}) = \mathrm{Mono}(\mathcal{C}) \cap \mathrm{Epi}(\mathcal{C})$$

subject to the following two standing assumptions:

1 $\mathrm{Iso}(\mathcal{C}) \subseteq \mathcal{B}$;

2 $\mathcal{B}$ is closed under composition.

The class $\mathcal{B}$ need not be closed under taking formal inverses, because a bimorphism need not be invertible.

Let $\mathrm{Sub}(\mathcal{C})$ denote the poset of full replete subcategories of $\mathcal{C}$, ordered by inclusion. Repleteness means closure under isomorphism and ensures that the definitions below depend only on categorical objects, not on chosen representatives.

**Definition 2.1.** *For $\mathcal{A} \in \mathrm{Sub}(\mathcal{C})$ define:*

$$S_{\mathcal{B}}(\mathcal{A}) = \big\{X \in \mathcal{C} \mid \textit{there exist } A \in \mathcal{A} \textit{ and } b : X \to A \textit{ in } \mathcal{B}\big\},$$
$$Q_{\mathcal{B}}(\mathcal{A}) = \big\{Y \in \mathcal{C} \mid \textit{there exist } A \in \mathcal{A} \textit{ and } b : A \to Y \textit{ in } \mathcal{B}\big\}.$$

*Both expressions denote the full replete subcategories generated by the displayed objects. Objects in $S_{\mathcal{B}}(\mathcal{A})$ are called $\mathcal{B}$-subobjects of objects of $\mathcal{A}$, while objects in $Q_{\mathcal{B}}(\mathcal{A})$ are called $\mathcal{B}$-factorobjects of objects of $\mathcal{A}$.*

The notation agrees with the subobject and factorobject operators used in the semireflexive-subcategory literature [3, 6]. Notice that the same morphism $b : X \to Y$ simultaneously presents $X$ as a $\mathcal{B}$-subobject of $Y$ and $Y$ as a $\mathcal{B}$-factorobject of $X$.

**Proposition 2.1.** *The maps $S_{\mathcal{B}}, Q_{\mathcal{B}} : \mathrm{Sub}(\mathcal{C}) \to \mathrm{Sub}(\mathcal{C})$ are closure operators: they are extensive, monotone, and idempotent.*

*Proof.* Because identity morphisms belong to $\mathcal{B}$, every object of $\mathcal{A}$ is both a $\mathcal{B}$-subobject and a $\mathcal{B}$-factorobject of itself; hence $\mathcal{A} \subseteq S_{\mathcal{B}}(\mathcal{A})$ and $\mathcal{A} \subseteq Q_{\mathcal{B}}(\mathcal{A})$. Monotonicity follows immediately from the definitions.

Suppose $X \in S_{\mathcal{B}}(S_{\mathcal{B}}(\mathcal{A}))$. Then there are morphisms $b_1 : X \to Y$ and $b_2 : Y \to A$ in $\mathcal{B}$, with $A \in \mathcal{A}$. By composition stability, $b_2 b_1 \in \mathcal{B}$, so $X \in S_{\mathcal{B}}(\mathcal{A})$. The reverse inclusion follows from extensivity, proving idempotence. The argument for $Q_{\mathcal{B}}$ is dual: if $A \to Y \to Z$ are in $\mathcal{B}$, their composite belongs to $\mathcal{B}$. ∎

**Definition 2.2.** *A full replete subcategory $\mathcal{A} \subseteq \mathcal{C}$ is called* $\mathcal{B}$-semireflexive *if*

$$S_{\mathcal{B}}(\mathcal{A}) = \mathcal{A} = Q_{\mathcal{B}}(\mathcal{A}).$$

*Equivalently, $\mathcal{A}$ is closed under both $\mathcal{B}$-subobjects and $\mathcal{B}$-factorobjects.*

Thus $\mathcal{B}$-semireflexive subcategories are the common fixed points of two closure operators. The following basic consequence will be refined in Section 3.

**Proposition 2.2.** *Arbitrary intersections of $\mathcal{B}$-semireflexive subcategories are $\mathcal{B}$-semireflexive. Consequently, every $\mathcal{A} \in \mathrm{Sub}(\mathcal{C})$ is contained in a least $\mathcal{B}$-semireflexive subcategory.*

*Proof.* Let $(\mathcal{A}_i)_{i\in I}$ be $\mathcal{B}$-semireflexive and set $\mathcal{A} = \bigcap_{i\in I} \mathcal{A}_i$. If $b : X \to Y$ belongs to $\mathcal{B}$ and $Y \in \mathcal{A}$, then $Y \in \mathcal{A}_i$ for every $i$, so $X \in \mathcal{A}_i$ for every $i$ because each $\mathcal{A}_i$ is closed under $\mathcal{B}$-subobjects. Hence $X \in \mathcal{A}$. Closure under $\mathcal{B}$-factorobjects is analogous. The least semireflexive subcategory containing a given $\mathcal{A}$ is the intersection of all semireflexive subcategories containing it. ∎

## 3. THE COMPONENT THEOREM

The two closure conditions in Definition 2.3 imply that membership in a semireflexive subcategory is invariant along every $\mathcal{B}$-morphism, independently of its direction. This observation leads to a complete classification.

Let $\Gamma_{\mathcal{B}}(\mathcal{C})$ be the undirected graph whose vertices are the isomorphism classes of objects of $\mathcal{C}$. Two vertices $[X]$ and $[Y]$ are joined by an edge whenever there is a morphism $X \to Y$ in $\mathcal{B}$ or a morphism $Y \to X$ in $\mathcal{B}$. Because all isomorphisms lie in $\mathcal{B}$, this graph is independent of the chosen representatives. Write

$$X \sim_{\mathcal{B}} Y$$

when $[X]$ and $[Y]$ belong to the same connected component of $\Gamma_{\mathcal{B}}(\mathcal{C})$. Equivalently, there is a finite zigzag

$$X = X_0 \;\leftrightarrow\; b_1 \;\; X_1 \;\leftrightarrow\; b_2 \;\; \cdots \;\leftrightarrow\; b_n \;\; X_n = Y, \qquad b_i \in \mathcal{B}, \tag{1}$$

where each arrow may point in either direction.

**Theorem 3.1** (Component theorem)**.** *For a full replete subcategory $\mathcal{A} \subseteq \mathcal{C}$, the following conditions are equivalent:*

1. *$\mathcal{A}$ is $\mathcal{B}$-semireflexive;*
2. *for every $b : X \to Y$ in $\mathcal{B}$, one has $X \in \mathcal{A}$ if and only if $Y \in \mathcal{A}$;*
3. *$\mathcal{A}$ is a union of $\sim_{\mathcal{B}}$-equivalence classes, or equivalently a union of connected components of $\Gamma_{\mathcal{B}}(\mathcal{C})$.*

*Proof.* Assume (i), and let $b : X \to Y$ be in $\mathcal{B}$. If $X \in \mathcal{A}$, closure under $\mathcal{B}$-factorobjects gives $Y \in \mathcal{A}$. If $Y \in \mathcal{A}$, closure under $\mathcal{B}$-subobjects gives $X \in \mathcal{A}$. Hence (ii) holds.

Condition (ii) propagates membership along each edge of $\Gamma_{\mathcal{B}}(\mathcal{C})$, and therefore along every finite zigzag of the form (1); this proves (iii).

Finally, assume (iii). If $b : X \to Y$ lies in $\mathcal{B}$, then $X \sim_{\mathcal{B}} Y$. Hence $Y \in \mathcal{A}$ implies $X \in \mathcal{A}$, so $S_{\mathcal{B}}(\mathcal{A}) \subseteq \mathcal{A}$, and $X \in \mathcal{A}$ implies $Y \in \mathcal{A}$, so $Q_{\mathcal{B}}(\mathcal{A}) \subseteq \mathcal{A}$. Extensivity of the two operators gives the reverse inclusions. Therefore $\mathcal{A}$ is $\mathcal{B}$-semireflexive. ∎

The theorem transforms a pair of closure conditions into a component-selection problem. Let

$$\pi_0^{\mathcal{B}}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C})/{\sim_{\mathcal{B}}}$$

denote the set of $\mathcal{B}$-components, understood up to isomorphism.

**Corollary 3.1.** *The poset* $\mathrm{SR}_{\mathcal{B}}(\mathcal{C})$ *of all* $\mathcal{B}$*-semireflexive full replete subcategories is canonically isomorphic to the powerset Boolean algebra*

$$\mathcal{P}\big(\pi_0^{\mathcal{B}}(\mathcal{C})\big).$$

*In particular,* $\mathrm{SR}_{\mathcal{B}}(\mathcal{C})$ *is a complete atomic Boolean algebra. Its atoms are the full replete subcategories generated by individual* $\mathcal{B}$*-components.*

*Proof.* By Theorem 3.1, a $\mathcal{B}$-semireflexive subcategory is uniquely determined by the set of components it contains, and every set of components determines such a subcategory. Inclusion, intersection, union, and complement correspond to the Boolean operations in the powerset. ∎

This Boolean structure is stronger than the complete-lattice conclusion obtained merely from intersections. In particular, joins are ordinary unions, because a union of component-saturated classes is again component-saturated.

**Definition 3.1.** *For* $\mathcal{A} \in \mathrm{Sub}(\mathcal{C})$ *define its* $\mathcal{B}$-semireflexive closure *by*

$$\mathcal{L}_{\mathcal{B}}(\mathcal{A}) = \big\{X \in \mathcal{C} \mid X \sim_{\mathcal{B}} A \textit{ for some } A \in \mathcal{A}\big\}.$$

**Corollary 3.2.** *The subcategory* $\mathcal{L}_{\mathcal{B}}(\mathcal{A})$ *is the least* $\mathcal{B}$*-semireflexive subcategory containing* $\mathcal{A}$*. Moreover,*

$$\mathcal{L}_{\mathcal{B}}(\mathcal{A}) = \bigcup_{A \in \mathrm{Ob}(\mathcal{A})} \mathcal{L}_{\mathcal{B}}(\{A\}),$$

*so the closure operator* $\mathcal{L}_{\mathcal{B}}$ *preserves arbitrary unions and is generated by singleton closures.*

*Proof.* The displayed class is a union of all components meeting $\mathcal{A}$, hence it is semireflexive by Theorem 3.1. Any semireflexive subcategory containing $\mathcal{A}$ must contain every component meeting $\mathcal{A}$, so it contains $\mathcal{L}_{\mathcal{B}}(\mathcal{A})$. The second assertion is immediate from the definition. ∎

**Remark 3.1.** *One may attempt to construct the least semireflexive closure by alternating the two operators $S_{\mathcal{B}}$ and $Q_{\mathcal{B}}$. The component formula is more precise: an object belongs to $\mathcal{L}_{\mathcal{B}}(\mathcal{A})$ exactly when it can be reached from $\mathcal{A}$ by a finite sequence of applications of $S_{\mathcal{B}}$ and $Q_{\mathcal{B}}$. No transfinite iteration is required.*

## 4. CHANGE OF THE BIMORPHISM CLASS AND CATEGORICAL INVARIANCE

Semireflexivity depends on the selected class of bimorphisms. Enlarging that class creates more edges, merges components, and imposes stronger closure conditions.

**Proposition 4.1** (Monotonicity)**.** *Let $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathrm{Bim}(\mathcal{C})$ satisfy the standing assumptions. Then*

$$\mathrm{SR}_{\mathcal{B}_2}(\mathcal{C}) \subseteq \mathrm{SR}_{\mathcal{B}_1}(\mathcal{C}) \quad \textit{and} \quad \mathcal{L}_{\mathcal{B}_1}(\mathcal{A}) \subseteq \mathcal{L}_{\mathcal{B}_2}(\mathcal{A})$$

*for every full replete subcategory $\mathcal{A}$.*

*Proof.* Every $\mathcal{B}_1$-edge is a $\mathcal{B}_2$-edge, so every $\mathcal{B}_1$-component is contained in a $\mathcal{B}_2$-component. A union of $\mathcal{B}_2$-components is therefore a union of $\mathcal{B}_1$-components, proving the first inclusion by Theorem 3.1. The second inclusion follows from the component description of closure. ∎

The next result shows that $\mathcal{B}$-semireflexivity is genuinely categorical rather than presentation-dependent.

**Theorem 4.1** (Invariance under equivalence)**.** *Let $F : \mathcal{C} \to \mathbb{D}$ be an equivalence of essentially small categories. Let $F\mathcal{B}$ be the replete class of morphisms of $\mathbb{D}$ isomorphic in the arrow category to images $F(b)$ with $b \in \mathcal{B}$. Then $F\mathcal{B}$ is a composition-stable class of bimorphisms containing all isomorphisms, and $F$ induces a Boolean-algebra isomorphism*

$$\mathrm{SR}_{\mathcal{B}}(\mathcal{C}) \cong \mathrm{SR}_{F\mathcal{B}}(\mathbb{D}).$$

*Furthermore,*

$$F\big(\mathcal{L}_{\mathcal{B}}(\mathcal{A})\big) \simeq \mathcal{L}_{F\mathcal{B}}\big(F(\mathcal{A})\big).$$

*Proof.* Equivalences preserve and reflect monomorphisms, epimorphisms, isomorphisms, and composition. Hence $F\mathcal{B}$ has the stated properties. The functor $F$ identifies isomorphism classes of objects and carries $\mathcal{B}$-edges to $F\mathcal{B}$-edges. A quasi-inverse carries $F\mathcal{B}$-edges back to $\mathcal{B}$-edges. Therefore $F$ induces a bijection between the component sets $\pi_0^{\mathcal{B}}(\mathcal{C})$ and $\pi_0^{F\mathcal{B}}(\mathbb{D})$. The conclusion follows from Corollary3.1 and Corollary3.2. ∎

In applications, the class $\mathcal{B}$ is often specified by a generating family rather than listed in full.

**Proposition 4.2.** *Let $\mathcal{G}$ be any class of bimorphisms, and let $\langle\mathcal{G}\rangle$ be the smallest class containing $\mathcal{G}$ and all isomorphisms and closed under composition. Then the connected components of $\Gamma_{\langle\mathcal{G}\rangle}(\mathcal{C})$ are exactly the connected components of the undirected graph generated by the morphisms in $\mathcal{G}$.*

*Proof.* Every $\mathcal{G}$-edge is an $\langle\mathcal{G}\rangle$-edge. Conversely, a morphism in $\langle\mathcal{G}\rangle$ is a finite composite of morphisms from $\mathcal{G}$ and isomorphisms. Its domain and codomain are therefore connected by a finite zigzag of $\mathcal{G}$-morphisms. Thus the two graphs have the same connected components. ∎

This result is useful when $\mathcal{B}$ is produced by units of reflections, completion morphisms, dense embeddings, or a factorisation system: semireflexive closure depends only on the undirected reachability generated by those morphisms.

## 5. REFLECTIVE SUBCATEGORIES AND SEMIREFLEXIVE PRODUCTS

We now connect the preceding construction with the reflective setting in which semireflexive subcategories were originally studied. Let $\mathcal{L}$ be a reflective full replete subcategory of $\mathcal{C}$, with reflector

$$\ell : \mathcal{C} \longrightarrow \mathcal{L}$$

and unit $\ell_X : X \to \ell X$. Following the established notation, define

$$\varepsilon_{\mathcal{L}} = \{e \in \mathrm{Epi}(\mathcal{C}) \mid \ell(e) \in \mathrm{Iso}(\mathcal{L})\}. \tag{2}$$

In categories of Hausdorff locally convex spaces considered in [3, 4, 6], the relevant classes $\varepsilon_{\mathcal{L}}$ consist of bimorphisms under the stated hypotheses. In the remainder of this section, we assume that $\varepsilon_{\mathcal{L}}$ contains all isomorphisms and is closed under composition, so it may play the role of $\mathcal{B}$.

**Definition 5.1.** *A full replete subcategory $\mathcal{A}$ is* $\mathcal{L}$-semireflexive *when it is closed under $\varepsilon_{\mathcal{L}}$-subobjects and $\varepsilon_{\mathcal{L}}$-factorobjects; equivalently, when it is $\varepsilon_{\mathcal{L}}$-semireflexive in the sense of Definition 2.2.*

The component theorem immediately yields a classification that does not appear explicitly in the definition.

**Corollary 5.1** (Reflector-component criterion)**.** *A full replete subcategory $\mathcal{A}$ is $\mathcal{L}$-semireflexive if and only if it is a union of connected components of the graph generated by the epimorphisms inverted by $\ell$.*

*Proof.* Apply Theorem 3.1 to $\mathcal{B} = \varepsilon_{\mathcal{L}}$. ∎

This statement separates two kinds of information. The reflector determines which epimorphisms become isomorphisms; graph connectivity then determines all $\mathcal{L}$-semireflexive subcategories.

We also recall the semireflexive product. If $\mathcal{L}$ is reflective and $\mathcal{A}$ is a full subcategory of $\mathcal{C}$, define

$$\mathcal{L} \times_{sr} \mathcal{A} = \{X \in \mathcal{C} \mid \ell X \in \mathcal{A}\}. \tag{3}$$

Thus an object is $(\mathcal{L}, \mathcal{A})$-semireflexive when its $\mathcal{L}$-reflection lies in $\mathcal{A}$. The construction and its applications to locally convex spaces were developed in [3, 6].

A classical criterion states that, for reflective subcategories $\mathcal{R}$ and $\mathcal{L}$ in the setting of the cited theory,

$$\mathcal{R} = \mathcal{L} \times_{sr} \mathcal{R} \quad \Longleftrightarrow \quad \mathcal{R} \text{ is } \mathcal{L}\text{-semireflexive}. \tag{4}$$

Combining this criterion with Corollary 5.1 gives the following reformulation.

**Corollary 5.2** (Component form of the semireflexive-product criterion)**.** *Under the hypotheses for which* (4) *holds, the following are equivalent for a reflective subcategory* $\mathcal{R}$*:*

*1* $\mathcal{R} = \mathcal{L} \times_{sr} \mathcal{R}$*;*

*2* $\mathcal{R}$ *is* $\varepsilon_{\mathcal{L}}$*-semireflexive;*

*3* $\mathcal{R}$ *is a union of* $\varepsilon_{\mathcal{L}}$*-components.*

*Proof.* The equivalence of (i) and (ii) is the semireflexive-product criterion from [4, 6]. The equivalence of (ii) and (iii) is Theorem 3.1. ∎

The corollary turns a universal-property statement involving a reflector into an objectwise saturation condition. It may therefore be useful in examples where the class $\varepsilon_{\mathcal{L}}$ is easier to recognise than the whole semireflexive product.

# 6. EXAMPLES

## 6.1. BALANCED CATEGORIES

**Example 1.** *Let* $\mathcal{C}$ *be balanced and take* $\mathcal{B} = \mathrm{Bim}(\mathcal{C})$*. Then* $\mathcal{B} = \mathrm{Iso}(\mathcal{C})$*. Every connected component of* $\Gamma_{\mathcal{B}}(\mathcal{C})$ *consists of a single isomorphism class, and every full replete subcategory of* $\mathcal{C}$ *is* $\mathcal{B}$*-semireflexive.*

This example shows that nontrivial semireflexive behaviour requires either a nonbalanced ambient category or a specially chosen class of morphisms that connects distinct isomorphism classes. Categories of sets, groups, modules, and many algebraic varieties are balanced, so the class of all bimorphisms yields no additional closure there. A reflector-specific class $\varepsilon_{\mathcal{L}}$ may nevertheless remain meaningful if it is not simply the class of all bimorphisms.

## 6.2. HAUSDORFF LOCALLY CONVEX SPACES

Let $\mathbf{LCS}_H$ denote the category of Hausdorff locally convex spaces and continuous linear maps. In this category, injective maps are monomorphisms and maps with dense range are epimorphisms; the interaction of such morphisms with reflective subcategories is central in the semireflexive literature [8, 11, 3].

**Example 2.** *The canonical inclusion*

$$j : \ell^1 \longrightarrow \ell^2$$

*is continuous because* $\|x\|_2 \leq \|x\|_1$ *for* $x \in \ell^1$. *It is injective, and its image is dense in* $\ell^2$ *because the finitely supported sequences lie in* $\ell^1$ *and are dense in* $\ell^2$. *Therefore* $j$ *is a bimorphism in* $\mathbf{LCS}_H$, *but it is not an isomorphism. For* $\mathcal{B} = \mathrm{Bim}(\mathbf{LCS}_H)$, *any* $\mathcal{B}$*-semireflexive subcategory contains* $\ell^1$ *if and only if it contains* $\ell^2$.

**Example 3** (Completion components)**.** *Let* $\Gamma_0$ *be the reflective subcategory of complete Hausdorff locally convex spaces, with completion reflector* $\gamma$. *For a Hausdorff locally convex space* $X$, *the unit*

$$\gamma_X : X \longrightarrow \widehat{X}$$

*is an injective map with dense image. Moreover, applying completion to* $\gamma_X$ *yields an isomorphism. Hence* $\gamma_X \in \varepsilon_{\Gamma_0}$ *in the standard completion setting. It follows from Corollary 5.1 that every* $\Gamma_0$*-semireflexive subcategory contains* $X$ *if and only if it contains its completion* $\widehat{X}$.

This example explains categorically why semireflexive closure often groups an incomplete space together with a canonical completion. More generally, whenever a reflection unit belongs to the chosen bimorphism class, the original object and its reflection lie in the same component.

## 6.3. DEPENDENCE ON THE CHOSEN CLASS

**Example 6.1.** *In* $\mathbf{LCS}_H$, *let* $\mathcal{B}_1 = \mathrm{Iso}(\mathbf{LCS}_H)$ *and let* $\mathcal{B}_2$ *be the composition-stable class generated by all completion embeddings. Then every full replete subcategory is* $\mathcal{B}_1$*-semireflexive, whereas a* $\mathcal{B}_2$*-semireflexive subcategory must be saturated under passage between* $X$ *and* $\widehat{X}$. *Since* $\mathcal{B}_1 \subseteq \mathcal{B}_2$, *Proposition 4.1 gives*

$$\mathrm{SR}_{\mathcal{B}_2}(\mathbf{LCS}_H) \subseteq \mathrm{SR}_{\mathcal{B}_1}(\mathbf{LCS}_H).$$

*Thus the choice of* $\mathcal{B}$ *controls the resolution at which subcategories can distinguish objects.*

## 7. FINITE CATEGORIES AND EFFECTIVE CLASSIFICATION

When $\mathcal{C}$ has finitely many isomorphism classes of objects and $\mathcal{B}$ is explicitly given, the component theorem gives a direct algorithm for enumerating all $\mathcal{B}$-semireflexive subcategories.

Construct an undirected graph with one vertex for each isomorphism class and an edge between $[X]$ and $[Y]$ whenever a $\mathcal{B}$-morphism connects representatives in either direction. Compute its connected components by depth-first search, breadth-first search, or a disjoint-set data structure.

**Proposition 7.1.** *Suppose $\Gamma_{\mathcal{B}}(\mathcal{C})$ has $k$ connected components. Then $\mathcal{C}$ has exactly $2^k$ full replete $\mathcal{B}$-semireflexive subcategories. If $\mathcal{A}$ is a class of objects, its semireflexive closure is the union of precisely those components meeting $\mathcal{A}$.*

*Proof.* The statement is an immediate consequence of the Boolean-algebra isomorphism in Corollary 3.1 and the closure formula in Corollary 3.2. ∎

This finite form is particularly useful for small categories arising from diagrams, finite model classes, or computer-assisted exploration. The categorical problem reduces to graph connectivity; no repeated verification of subobject and factorobject conditions is needed once the components have been computed.

## 8. FURTHER STRUCTURAL CONSEQUENCES

The component description yields several additional observations.

**Proposition 8.1.** *A nonempty $\mathcal{B}$-semireflexive subcategory $\mathcal{A}$ is an atom of $\mathrm{SR}_{\mathcal{B}}(\mathcal{C})$ if and only if any two objects of $\mathcal{A}$ are connected by a finite $\mathcal{B}$-zigzag.*

*Proof.* Atoms of the powerset algebra in Corollary 3.1 are singleton sets of components. Translating back to subcategories gives the assertion. ∎

**Proposition 8.2.** *Let $F : \mathcal{C} \to \mathbb{D}$ be a functor such that $F(\mathcal{B}) \subseteq \mathcal{E}$, where $\mathcal{E}$ is a composition-stable class of bimorphisms in $\mathbb{D}$ containing all isomorphisms. Then $F$ sends each $\mathcal{B}$-component into a single $\mathcal{E}$-component. Consequently,*

$$F\big(\mathcal{L}_{\mathcal{B}}(\mathcal{A})\big) \subseteq \mathcal{L}_{\mathcal{E}}\big(F(\mathcal{A})\big).$$

*Proof.* The image of a finite $\mathcal{B}$-zigzag is a finite $\mathcal{E}$-zigzag. Hence $\mathcal{B}$-connected objects have $\mathcal{E}$-connected images. The closure inclusion follows. ∎

Unlike Theorem 4.1, the inclusion may be strict because a general functor need not reflect connectivity. This distinction is relevant when comparing semireflexive structures across forgetful functors or embeddings of categories.

**Remark 8.1.** *The theory depends only on the symmetric reachability relation generated by $\mathcal{B}$, not on the orientation or internal factorisation of individual morphisms. Reflective information enters through the selection of $\varepsilon_{\mathcal{L}}$; once that class is fixed, all semireflexive subcategories are determined by its components. This division suggests a two-stage research strategy: first identify the morphisms inverted by the reflector, then analyse their connectivity.*

## 9. CONCLUSIONS AND OPEN DIRECTIONS

We have developed a general closure theory for subcategories that are simultaneously closed under subobjects and factorobjects determined by a class $\mathcal{B}$ of bimorphisms. The subobject and factorobject assignments are closure operators, but their common fixed points admit a substantially sharper description: they are exactly unions of connected components generated by $\mathcal{B}$-morphisms. This yields a complete atomic Boolean algebra of $\mathcal{B}$-semireflexive subcategories and an explicit finite-zigzag formula for semireflexive closure.

The component viewpoint is stable under equivalence of categories and makes the dependence on the morphism class transparent. In balanced categories, the class of all bimorphisms produces only isomorphism components, so every full replete subcategory is semireflexive. In the nonbalanced category of Hausdorff locally convex spaces, dense linear embeddings create larger components and nontrivial closure conditions. For reflective subcategories, taking $\mathcal{B} = \varepsilon_{\mathcal{L}}$ converts $\mathcal{L}$-semireflexivity into saturation under morphisms inverted by the reflector and gives a component form of the semireflexive-product criterion.

Several questions remain relevant for the category of Hausdorff locally convex spaces. One may seek explicit descriptions of the $\varepsilon_{\mathcal{L}}$-components associated with classical reflectors, determine when these components admit canonical representatives, and study how component partitions change under inclusion, intersection, or semireflexive products of reflective subcategories. Another direction is the interaction with factorisation systems: stability under pullbacks, pushouts, or products may impose additional structure on the component Boolean algebra. Finally, finite or finitely presented categorical models invite computational implementations of semireflexive saturation and comparison of reflector-generated morphism classes.

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# ON THE FUNCTORIAL ISOMORPHISM BETWEEN THE LATTICES OF $\mathcal{P}$-QUOTIENTS AND OF SATURATED SETS OF MORPHISMS

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**Abstract** Let $\mathcal{C}$ be a small, locally and colocally category with products, $(\mathcal{P}, \mathcal{I})$ a factorization structure in category $\mathcal{C}$, $\mathcal{R}$ a $\mathcal{P}$-reflective subcategory and $B$ - an $\mathcal{I}$-cogenerator of the subcategory $\mathcal{R}$. For every object $X$ we study two ordered structures: the class $\mathbb{P}(X)$ of $\mathcal{P}$-morphisms $p : X \to Y$ with $Y \in |\mathcal{R}|$, and the class $\mathcal{S}(X)$ of saturated subsets of $\mathcal{C}(X, B)$. We prove that, under the certain conditions $PS$ (for any morphism $p : X \to Z \in \mathcal{P}$ and any morphism $f : Y \to Z \in \mathcal{C}$ there exists the pushout square built on these morphisms) or $PD$ (for any morphism $p : Z \to X \in \mathcal{P}$ and any morphism $f : Z \to Y \in \mathcal{C}$ there exists the pullback square built on these morphisms), both are complete lattices and that the assignments $X \to \mathbb{P}(X)$ and $X \to \mathcal{S}(X)$ extend to covariant functors $\mathcal{F}$, $\mathcal{S} : \mathcal{C} \to \mathcal{L}at$, as well as to contravariant functors $\mathcal{F}^c$, $\mathcal{S}^c : \mathcal{C} \to \mathcal{L}at$. The main results establish that the functorial morphisms $\alpha : \mathcal{S} \to \mathcal{F}$, $\beta : \mathcal{F} \to \mathcal{S}$ and their contravariant analogues $\overline{\alpha} : \mathcal{S}^c \to \mathcal{F}^c$ and $\overline{\beta} : \mathcal{F}^c \to \mathcal{S}^c$ are mutually inverse, yielding functorial isomorphisms $\mathcal{F} \approxeq \mathcal{S}$ and $\mathcal{F}^c \approxeq \mathcal{S}^c$. Thus, the reflective quotients admit two equivalent descriptions: an external one, by morphisms of the class $\mathbb{P}$, and an internal one, by saturated families of morphisms into the cogenerator $B$. Examples in the categories of sets, zero-dimensional spaces and Tikhonov spaces illustrate the results.

**Keywords:** factorization structure, reflective subcategory, cogenerator, complete lattice, functorial isomorphism, saturated set.

**2020 MSC:** 46 M 15, 18 F 60.

## 1. THE COVARIANT FUNCTOR $\mathcal{F} : \mathcal{C} \to \mathcal{L}at$

**1.1.** Standing assumptions on the category $\mathcal{C}$:

- $\mathcal{C}$ - a small locally and colocally category with products;
- $(\mathcal{P}, \mathcal{I})$ - the factorization structure in category $\mathcal{C}$;
- $\mathcal{R}$ - the $\mathcal{P}$-reflective subcategory in category $\mathcal{C}$;
- $B$ - an $\mathcal{I}$-cogenerator to the subcategory $\mathcal{R}$: $\mathcal{R} \subset \mathcal{S}_I(\mathbb{B}^\tau | \tau$ - a cardinal).

In some statements we will assume that one of the following two conditions holds:

$PS$ - for any morphism $p : X \to Z \in \mathcal{P}$ and any morphism $f : Y \to Z \in \mathcal{C}$ there exists the pushout square built on these morphisms;

$PD$ - for any morphism $p : Z \to X \in \mathcal{P}$ and any morphism $f : Z \to Y \in \mathcal{C}$ there exists the pullback square built on these morphisms.

**Remark.** 1. *The conditions $PS$ and $PD$ do not always hold. For example, in the category of Tikhonov spaces $Th$, let $f : X \to Y$ be an epi for which $f(X) \neq Y$, and $g : F \to Y$ and $g(F) \not\subseteq f(X)$, where $F$ is the space of a point. Then in the category of sets the pushout square constructed on the morphisms $(f, g)$ is the set $\varnothing$ with the mappings $\varnothing : \varnothing \to X$, $\varnothing : \varnothing \to Y$. In the category $Th$ the pushout square does not exist.*

2. *In concrete categories, if $\mathcal{P} \subset \mathcal{E}_u$, where $\mathcal{E}_u$ is the class of surjective morphisms, the conditions $PS$ and $PD$ hold.*

**1.2.** Let $F$ be the final object of the category $\mathcal{C}$ (as the product of an empty set of objects) and $f^X : X \to F$ the unique morphism from $X$ to $F$.

**Definition.** *The morphism $g : X \to Y$ is called constant if*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f^X"'] & & Y \\ & F \arrow[ur, dashed, "u"'] & \end{tikzcd}$$

$$g = u \cdot f^X$$

*for a morphism $u : F \to Y$.*

We denote by $Const(X, Y)$ the set of all constant morphisms from the object $X$ into the object $Y$.

*A category is called connex if $Hom(X, Y) \neq \varnothing$ for any of its objects $X$ and $Y$.*

*For $X \in |\mathcal{C}|$ we denote by $\mathbb{P}(X)$ the class of all morphisms $p : X \to Y$ that belong to the class $\mathcal{P}$ and $Y \in |\mathcal{R}|$.*

**1.3. Lemma.** *Let $\mathcal{C}$ be a connex category. Then $\mathbb{P}(X)$ is a complete lattice with the smallest element $f^X : X \to F$ and the largest element $r^X : X \to rX$.*

In a connex category $f^X$ is retractible, and any morphism $g : F \to X$ is sectionable: $f^X \cdot g = 1$. Thus $f^X \in P$ and $F \in |\mathcal{R}|$. Hence $f^X \in \mathbb{P}(X)$.

Now let $\{p_\alpha : X \to Y_\alpha | \alpha \in \mathcal{A}\}$ be a family of elements of the set $\mathbb{P}(X)$, $Y = \prod\{Y_\alpha | \alpha \in \mathcal{A}\}$ with the canonical projections $\{q_\alpha : Y \to Y_\alpha\}$, $w : X \to Y$ the canonical morphism with the properties

$$q_\alpha \cdot w = p_\alpha, \alpha \in \mathcal{A},$$

and

$$w = i \cdot p$$

the $(\mathcal{P}, \mathcal{I})$-factorization of w:

$$\begin{tikzcd}
Z \arrow[r, dashed, "i"] & Y \arrow[d, "q_\alpha"] \\
X \arrow[u, dashed, "p"] \arrow[ur, dashed, "w"] \arrow[r, dashed, "p_\alpha"'] & Y_\alpha
\end{tikzcd}$$

Since $Y \in |\mathcal{R}|$ and $i \in \mathcal{I}$, it follows that $Z \in |\mathcal{R}|$. It is easy to verify that

$$p = sup\{p_\alpha, \alpha \in \mathcal{A}\}.$$

**1.4. The ordered set $\mathbb{P}(X)$.**

1. Let $r : X \to rX$ be an $\mathcal{R}$ -replica of the object $X$. Then $r^X$ is the largest element of the ordered set $\mathbb{P}(X)$.

2. Let $p : X \to Y \in \mathcal{P}$ and $r^Y : Y \to rY$ be an $\mathcal{R}$ -replica of $Y$. Then $r^Y \cdot p \in \mathbb{P}(X)$.

**1.5. Lemma**. *Let the category $\mathcal{C}$ possess the properties $PS$ and $PD$. Then $\mathbb{P}(X)$ is a complete lattice.*

*Proof.* $Min(p_1, p_2)$. Let $p_1 : X \to X_1$ and $p_2 : X \to X_2$ be elements of the set $\mathbb{P}(X)$,

$$p_2' \cdot p_1 = p_1' \cdot p_2$$

the pullback square built on the morphisms $p_1$ and $p_2$

$$\begin{tikzcd}
X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X_1 \arrow[d, dashed, "p_2'"] & \\
X_2 \arrow[r, dashed, "p_1'"] & Y \arrow[r, dashed, "r^Y"] & rY
\end{tikzcd}$$

$r^Y : Y \to rY$ $\mathcal{R}$-replica of $Y$. Then $p_1$, $p_2 \in \mathcal{P}$ and $r^Y \cdot p_2' \cdot p_1 = min(p_1, p_2)$.

$Sup(p_1, p_2)$. Let us return to the previous diagram. Let

$$p_1' \cdot p_2'' = p_2' \cdot p_1''$$

the pushout square built on the morphisms $p_1'$ and $p_2'$. Then

$$p_1 = p_1'' \cdot u,$$

$$p_2 = p_2'' \cdot u$$

for some $u$. Let

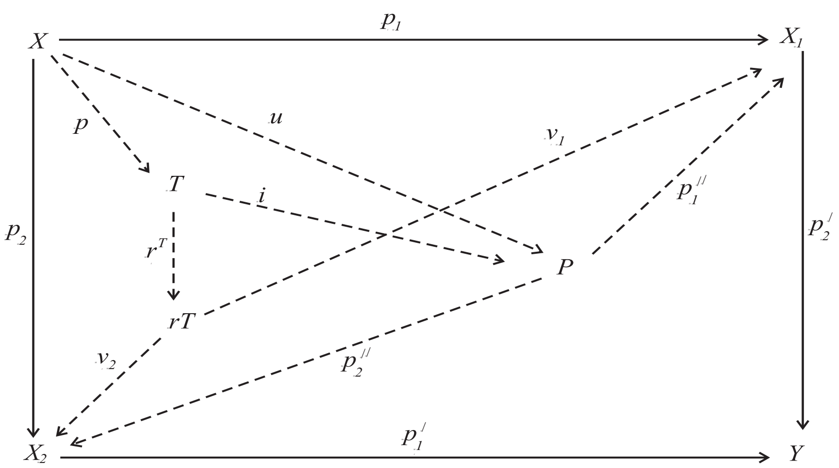
$$u = i \cdot p$$

the $(\mathcal{P}, \mathcal{I})$-factorization of the respective morphism and $r^T : T \to rT$ be the $\mathcal{R}$ -replica of $T$. Since $X_1,\ X_2 \in |\mathcal{R}|$, we deduce that

$$p_1'' \cdot i = v_1 \cdot r^T,$$

$$p_2'' \cdot i = v_2 \cdot r^T$$

for two morphisms $v_1$ and $v_2$. Then




$$p_1 = v_1 \cdot r^T \cdot p,$$

$$p_2 = v_2 \cdot r^T \cdot p$$

and $r^T \cdot p \in \mathbb{P}(X)$ that $p \geq p_1$, $p \geq p_2$. One verifies that $p = sup(p_1, p_2)$. ∎

**1.6. The functor** $\mathcal{F} : \mathcal{C} \to \mathcal{L}at$**.** On objects $\mathcal{F}(X) = \mathbb{P}(X)$ for $X \in |\mathcal{C}|$. Let $u : X \to Y$. On the morphisms $u$ and $r^X$ we construct the pullback squares:

$$u' \cdot r^X = t \cdot u,$$

$$u' \cdot r(v) = t' \cdot u'.$$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "r^X"'] \arrow[dd, bend right, "v"'] & Y \arrow[d, "t"] & \\
rX \arrow[r, "u'"] \arrow[d, dashed, "r(v)"'] & T \arrow[r, "r^T"] \arrow[d, "t'"] & rT \arrow[d, dashed, "r(t')"] \\
A \arrow[r, "u'"] & T' \arrow[r, "r^{T'}"] & rT'
\end{tikzcd}$$

Let $v : X \to A \in \mathcal{F}(X)$. Define

$$\mathcal{F}(u)(v) = r^{T'} \cdot t' \cdot t.$$

**Proposition.** *Let the category $\mathcal{C}$ possess the property $PD$ and $f : X \to Y \in \mathcal{C}$. Then $\mathcal{F}(f)$ preserves the maximal element of the lattice $\mathcal{F}(X)$: $\mathcal{F}(f)(r^X) = r^Y$.*

*Proof.* Let $r^X : X \to rX$ and $r^Y : Y \to rY$ be the $\mathcal{R}$-replicas of the respective objects, and

$$f' \cdot r^X = p' \cdot f,$$

the pullback square built on the morphisms $r^X$ and $f$. Then

$$r(f) \cdot r^X = r^Y \cdot f.$$

Thus

$$r(f) = t \cdot f',$$

$$r^Y = t \cdot p'.$$

for a morphism $t : T \to rY$. From the previous equality it follows that $t \in \mathcal{E}pi$ and it is easy to verify that $t$ is the $\mathcal{R}$-replica of the object $T$. So

$$\mathcal{F}(f)r^X = t \cdot p' = r^Y.$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r^X"'] & Y \arrow[d, dashed, "p'"] \arrow[dr, "r^Y"] & \\
rX \arrow[r, dashed, "f'"] \arrow[rr, dashed, bend right, "r(f)"'] & T \arrow[r, dashed, "t"] & rY
\end{tikzcd}$$

Consider $f : X \to Y$, $g : Y \to Z$ and $p : X \to A \in \mathcal{F}(X)$.

We construct the pullback squares:
on the morphisms $p$ and $f$ - $f' \cdot p = p' \cdot f$, (1)
on the morphisms $p'$ and $g$ - $g' \cdot p' = p'' \cdot g$, (2)
on the morphisms $r_1$ and $g'$ - $g'' \cdot r_1 = r_1' \cdot g'$, (3)
where $r_1 : P_1 \to rP_1$ is the $\mathcal{R}$-replica of the object $P_1$.

Further, let $r_2 : P_2 \to rP_2$ and $r_3 : P_3 \to rP_3$ be the $\mathcal{R}$-replicas of the respective objects. Then

$$r_2 \cdot g' = u \cdot r_1, \tag{4}$$

for a morphism $u$. Since (3) is a pullback square, from equality (4) it follows that there exists a morphism $u_1 : P_3 \to rP_2$ such that

$$r_2 = u_1 \cdot r_1', \tag{5}$$

$$u = u_1 \cdot g''. \tag{6}$$

Further,

$$u_1 = u_2 \cdot r_3. \tag{7}$$

The morphism $r_3 \cdot r_1'$ extends by the morphism $r_2$:

$$r_3 \cdot r_1' = v \cdot r_2. \tag{8}$$

We have

$$v \cdot u_2 \cdot r_3 \cdot r_1' = (from(7)) = v \cdot u_1 \cdot r_1' = (from(5)) = v \cdot r_2 = (from(8)) = r_3 \cdot r_1'$$

i.e.

$$v \cdot u_2 \cdot r_3 \cdot r_1' = r_3 \cdot r_1'. \tag{9}$$

Since $r_3 \cdot r_1'$ is an *epi*, we obtain

$$v \cdot u_2 = f. \tag{10}$$

From (5) we have $u_1 \in \mathcal{P}$, and from (7) - $u_2 \in \mathcal{P}$. So

$$u_2 = v^{-1}, \tag{11}$$

or

$$r_3 \cdot r_1' \cdot p'' \sim r_2 \cdot p''. \tag{12}$$

We have

$$(g' \cdot f') \cdot p = p'' \cdot (g \cdot f).$$

is the pullback square built on the morphisms $g \cdot f$ and $p$, and

$$g'' \cdot (r_1 \cdot p') = (r_1 \cdot p'') \cdot g$$

is the pullback square built on the morphisms $r_1 \cdot p'$ and $g$. So

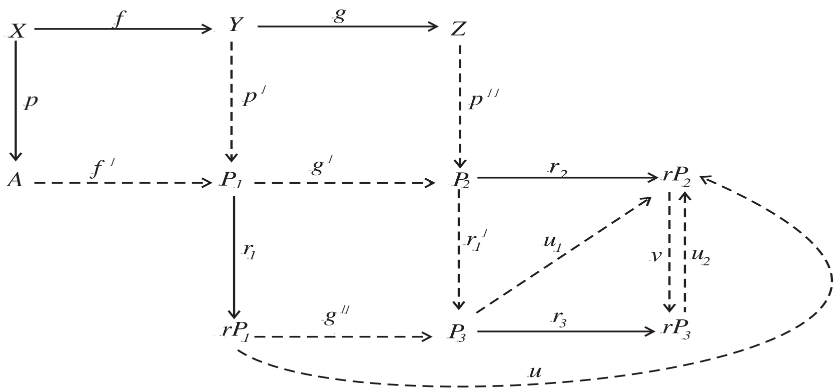
$$\mathcal{F}(g \cdot f)(p) = r_2 \cdot p'',$$

and

$$\mathcal{F}(g)(\mathcal{F}(f)(p)) = \mathcal{F}(g)(r_1 \cdot p') = r_3 \cdot r_1' \cdot p''.$$

Taking into account (12) we can say that

$$\mathcal{F}(g \cdot f) = \mathcal{F}(g) \cdot \mathcal{F}(f).$$




∎

## 2. THE CONTRAVARIANT FUNCTOR $\mathcal{F}^C : \mathcal{C} \to \mathcal{L}at$

On objects, this functor is defined in the same way as the covariant functor $\mathcal{F}$:

$$\mathcal{F}^c(X) = \mathcal{F}(X), X \in |\mathcal{C}|.$$

Let us define the functor $\mathcal{F}^c$ on morphisms. Let $f : X \to Y \in \mathcal{C}$ and $p : Y \to T \in \mathcal{P}$, $T \in |\mathcal{R}|$ be. We examine the $(\mathcal{P}, \mathcal{I})$-factorization of morphism $p \cdot f$:

$$p \cdot f = i_1 \cdot p_1.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "p_1"] & Y \arrow[d, dashed, "p"] \\ K \arrow[r, dashed, "i_1"] & T \end{tikzcd}$$

Then $p_1 \in \mathcal{P}$ and since $T \in |\mathcal{R}|$ and $\mathcal{R}$ is $\mathcal{P}$-reflective, we deduce that $K \in |\mathcal{R}|$. Thus $p_1 \in \mathcal{F}^c(X)$. By definition we establish:

$$\mathcal{F}^c(f)(p) = p_1.$$

Let $f : X \to Y \in \mathcal{C}$ and $g : Y \to Z$, $p \in \mathcal{F}^c(Z)$, and

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "p_2"] & Y \arrow[r, "g"] \arrow[d, dashed, "p_1"] & Z \arrow[d, "p"] \\ A_2 \arrow[r, dashed, "i_2"] & A_1 \arrow[r, dashed, "i_1"] & T \end{tikzcd}$$

$$p \cdot g = i_1 \cdot p_1,$$

$$p_1 \cdot f = i_2 \cdot p_2.$$

$(\mathcal{P}, \mathcal{I})$-factorization of respective morphisms. Then

$$p \cdot (g \cdot f) = (i_1 \cdot i_2) \cdot p_2$$

is the $(\mathcal{P}, \mathcal{I})$-factorization of morphism $p \cdot (g \cdot f)$. Thus

$$\mathcal{F}^c(g \cdot f)(p) = p_2,$$

and

$$\mathcal{F}^c(f)(\mathcal{F}^c(g)(p)) = \mathcal{F}^c(f)(p_1) = p_2.$$

So

$$\mathcal{F}^c(g \cdot f) = \mathcal{F}^c(f) \cdot \mathcal{F}^c(g).$$

## 3. THE COVARIANT FUNCTOR $\mathcal{S} : \mathcal{C} \to \mathcal{L}at$

**3.1.** Let $\mathcal{U} \subset \mathcal{C}(X, B)$ and let $w^{\mathcal{U}} : X \to B^{\mathcal{U}}$ be the canonical morphism with the property

$$f = q_f \cdot w^{\mathcal{U}}$$

for any $f \in \mathcal{U}$, where $q_f$ is the canonical projection.

$$\begin{tikzcd} X \arrow[rr, "w^{\mathcal{U}}"] \arrow[dr, "f"'] & & B^{\mathcal{U}} \arrow[dl, dashed, "q_f"] \\ & B & \end{tikzcd}$$

We examine the $(\mathcal{P}, \mathcal{I})$-factorization of this morphism:

$$\begin{tikzcd} X \arrow[r, "p^{\mathcal{U}}"] & t(\mathcal{U}) \arrow[r, "i^{\mathcal{U}}"] & B^{\mathcal{U}} \end{tikzcd}$$

For $p : X \to Y \in \mathcal{P}$ we denote by $\mathcal{E}xt_B(p)$ the set of all morphisms $f : X \to B$ extending through $p$. Thus with the notations above we have:

$$\mathcal{U} \subset \mathcal{E}xt_B(p^{\mathcal{U}}).$$

**Definition.** *The set $\mathcal{U} \subset C(X, B)$ is called saturated, if $t(\mathcal{U}) \in |\mathcal{R}|$ and $\mathcal{U} = \mathcal{E}xt_B(p^{\mathcal{U}})$.*

*By definition let $S(X)$ be the class of all saturated sets in $C(X, B)$, and $\mathbb{P}(X)$ the class of all morphisms $p : X \to Y$ belonging to the class $\mathcal{P}$ and $Y \in |\mathcal{R}|$.*

**3.2. Theorem.** *Let $X$ be an object of the category $\mathcal{C}$.*

1. *$Const(X, B)$ is the smallest element of the set $S(X)$.*
2. *Let category $\mathcal{C}$ be connex. If $\mathcal{U} = Const(X, B)$, then $p^{\mathcal{U}} = f^X$.*

*Proof.* 1. Let $\mathcal{V} \subset S(X)$ be and we will demonstrate that $Const(X, B) \subset \mathcal{V}$. For $g : X \to B \in \mathcal{V}$ let

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f^X"'] & & B \\ & F \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

$$g = h \cdot f^X.$$

We have the following diagram:

$$\begin{tikzcd}
t(\mathcal{V}) \arrow[r, "i^{\mathcal{V}}"] & B^{\mathcal{V}} \arrow[d, dashed, "f^{B^{\mathcal{V}}}"] & \\
X \arrow[u, "p^{\mathcal{V}}"] \arrow[ur, dashed, "w^{\mathcal{V}}"] \arrow[r, "f^X"] \arrow[rr, bend right, "g"'] & F \arrow[r, "h"] & B
\end{tikzcd}$$

Then

$$f^X = f^{B^{\mathcal{V}}} \cdot i^{\mathcal{V}} \cdot p^{\mathcal{V}}$$

or

$$g = (h \cdot f^X) = f^B \cdot i^{\mathcal{V}} \cdot p^{\mathcal{V}}$$

and since $\mathcal{V}$ is a saturated set, we deduce that $g \in \mathcal{V}$. Thus we have proved that $Const(X, B) \subset \mathcal{V}$, if $\mathcal{V} \in S(X)$.

2. Let us check that $p^{\mathcal{U}} = f^X$ for $\mathcal{U} = Const(X, B)$.

$$\begin{tikzcd}
F \arrow[r, dashed, "l_g"] \arrow[dr, dashed, "d^{\mathcal{U}}"] & B \\
X \arrow[u, "f^X"] \arrow[ur, "g"] \arrow[r, dashed, "w^{\mathcal{U}}"] & B^{\mathcal{U}} \arrow[u, dashed, "q_g"']
\end{tikzcd}$$

Let $g \in \mathcal{U}$. Then

$$g = l_g \cdot f^X$$

for a morphism $l_g$. There is a morphism $d^{\mathcal{U}}$ such that:

$$l_g = q_g \cdot d^{\mathcal{U}},$$

where $q_g$ are the canonical projections. Since the category $\mathcal{C}$ is connex, $f^X$ is rectactable and $l_g$ is sectionable. Thus $f^X \in \mathcal{P}$, $d^{\mathcal{U}} \in \mathcal{I}$, and

$$w^{\mathcal{U}} = d^{\mathcal{U}} \cdot f^X$$

is the $(\mathcal{P}, \mathcal{I})$-factorization of the morphism $w^{\mathcal{U}}$. We mention that $F \in |\mathcal{R}|$. ∎

**3.3. Definition.** *Let $\mathcal{M}$ be a class of mono of the category $\mathcal{C}$ and $\mathcal{A}$ a subcategory of it. The object $Z$ is called an $\mathcal{M}$-cogenerator for the subcategory*

*$\mathcal{A}$, if every element of it is an $\mathcal{M}$-subobject of an object of type $Z^\tau$, where $\tau$ is a cardinal:*

$$\mathcal{A} \subset \mathcal{S}_{\mathcal{M}}\{Z^\tau | \tau - cardinal\}.$$

*Examples.* 1. $\mathcal{D} = \{0, 1\}$ is an $\mathcal{M}_f$ -cogenerator for the subcategories of zero-dimensional and compact spaces.

2. $\mathcal{D} = \{0, 1\}$ is an $\mathcal{M}_p$ -cogenerator for the subcategory of zero-dimensional spaces.

3. $I = [0, 1]$ and $\mathbb{R}$ (the set of real numbers) is an $\mathcal{M}_f$ -cogenerator and an $\mathcal{M}_p$ -cogenerator for the subcategory $\mathcal{C}omp$ of compact spaces.

4. $I = [0, 1]$ and $\mathbb{R}$ is an $\mathcal{M}_p$ -cogenerator for the subcategory $\mathcal{Q}$ of *Hewitt* spaces.

**3.4.** Let $B \in |\mathcal{R}|$ and $B$ be $\mathcal{I}$-cogenerators for the subcategory $\mathcal{R}$: $\mathcal{R} \subset \mathrm{S}_{\mathcal{I}}\{B^\tau | \tau - cardinal\}$. Further, $p : X \to A \in \mathcal{P}$, $A \in |\mathcal{R}|$, $\mathcal{U} = \mathcal{E}xt_B(p)$, and

$$w^{\mathcal{U}} = i^{\mathcal{U}} \cdot p^{\mathcal{U}} \tag{13}$$

is the $(\mathcal{P}, \mathcal{I})$-factorization of the canonical morphism $w^{\mathcal{U}} : X \to B^{\mathcal{U}}$.

**Theorem.** *Let $(\mathcal{P}, \mathcal{I})$ be a factorization structure in the category $\mathcal{C}$, $B$ an $\mathcal{I}$-cogenerator of the reflective subcategory $\mathcal{R}$, $q : X \to Y \in \mathcal{S}(X)$ and $\mathcal{U} = \mathcal{E}xt_B(q)$. Then the factorobjects $p$ and $p^{\mathcal{U}}$ are isomorphic: $q \sim p^{\mathcal{E}xt_B(q)}$.*

*Proof.* Let $q_\alpha : B^{\mathcal{U}} \to B$ be a canonical projection. Then $q_\alpha \cdot w^{\mathcal{U}} \in \mathcal{U}$. So

$$q_\alpha \cdot w^{\mathcal{U}} = s_\alpha \cdot p(\forall \alpha) \tag{14}$$

for an $s_\alpha$. Thus

$$s_\alpha = q_\alpha \cdot s \tag{15}$$

for an $s$. We have

$$q_\alpha \cdot s \cdot p = (from(15)) = s_\alpha \cdot p = (from(14)) = q_\alpha \cdot w^{\mathcal{U}} = (from(15)) = q_\alpha \cdot i^{\mathcal{U}} \cdot p^{\mathcal{U}},$$

i.e.

$$q_\alpha \cdot s \cdot p = q_\alpha \cdot i^{\mathcal{U}} \cdot p^{\mathcal{U}}, (\forall \alpha),$$

or

$$s \cdot p = i^{\mathcal{U}} \cdot p^{\mathcal{U}}. \tag{16}$$

Since $p \in \mathcal{P}$ and $i^{\mathcal{U}} \in \mathcal{I}$ ($p \perp i$) there exists a diagonal $h : A \to t(\mathcal{U})$ with the properties:

$$p^{\mathcal{U}} = h \cdot p, \tag{17}$$

$$s = i^{\mathcal{U}} \cdot h.$$

Further, since $A \in |\mathcal{R}|$, there exists a morphism $i : A \to B^{\tau} \in \mathcal{I}$. Let $m_{\beta} : B^{\tau} \to B$ be a canonical projection. Then $m_{\beta} \cdot i \cdot p \in \mathcal{U}$ and there exists a canonical projection $n_{\beta} : B^{\mathcal{U}} \to B$ such that

$$m_{\beta} \cdot i \cdot p = n_{\beta} \cdot w^{\mathcal{U}}. \tag{18}$$

Then there exists the morphism $m : t(\mathcal{U}) \to B^{\tau}$ such that

$$n_{\beta} \cdot i^{\mathcal{U}} = m_{\beta} \cdot m. \tag{19}$$

We have

$$m_{\beta} \cdot i \cdot p = (from(18)) = n_{\beta} \cdot i^{\mathcal{U}} \cdot p^{\mathcal{U}} = (from(19)) = m_{\beta} \cdot m \cdot p^{\mathcal{U}},$$

i.e.

$$m_{\beta} \cdot i \cdot p = m_{\beta} \cdot m \cdot p^{\mathcal{U}}, (\forall \beta)$$

or

$$i \cdot p = m \cdot p^{\mathcal{U}}$$

with $p^{\mathcal{U}} \in \mathcal{P}$ and $i \in \mathcal{I}$ $(p^{\mathcal{U}} \perp i)$. Therefore

$$p = k \cdot p^{\mathcal{U}}, \tag{20}$$

$$i \cdot k = m.$$

Finally

$$k \cdot h \cdot p = (from(17)) = k \cdot p^{\mathcal{U}} = (from(20)) = p,$$

i.e.

$$k \cdot h \cdot p = p,$$

or

$$k \cdot h = 1.$$

Also

$$h \cdot k \cdot p^{\mathcal{U}} = (from(20)) = h \cdot p = (from(17)) = p^{\mathcal{U}},$$

i.e.

$$h \cdot k \cdot p^{\mathcal{U}} = p^{\mathcal{U}},$$

or

$$h \cdot k = 1.$$

$$\begin{tikzcd}
X \arrow[r, "p^{\mathcal{U}}"] \arrow[d, dash, "p"'] & t(\mathcal{U}) \arrow[d, "i^{\mathcal{U}}"] \arrow[dl, dashed, shift left, "k"] \arrow[ddl, dashed, "m"] \\
A \arrow[r, dashed, "s"] \arrow[ur, dashed, shift left, "h"] \arrow[d, dashed, "i"'] \arrow[dr, dashed, "s_\alpha"] & B^{\mathcal{U}} \arrow[d, dashed, shift right, "n_\beta"'] \arrow[d, shift left, "q_\alpha"] \\
B^{\tau} \arrow[r, "m_\beta"] & B
\end{tikzcd}$$

Thus we have proved that $k = h^{-1}$ and $p \sim p^{\mathcal{U}}$. ∎

**3.5. Theorem.** *Let $B$ be an $\mathcal{I}$-cogenerator of the subcategory $\mathcal{R}$. Then:*
1. *$S(X)$ is a complete lattice for any object $X$ of the category $\mathcal{C}$.*
2. *The map $S$ is a covariant functor $S : \mathcal{C} \to \mathcal{L}at$.*

*Proof.* 1. Let $\{\mathcal{U}_\alpha | \alpha \in \mathcal{A}\}$ be a family of elements of the set $S(X)$,

$$i_\alpha \cdot p_\alpha = w^{\mathcal{U}_\alpha}$$

the $(\mathcal{P}, \mathcal{I})$-factorization of respective morphisms, where

$$\begin{tikzcd}
X \arrow[r, "p_\alpha"] & Z_\alpha \arrow[r, "i_\alpha"] & B^{\mathcal{U}_\alpha}
\end{tikzcd}$$

If $Z = \prod\{Z_\alpha | \alpha \in \mathcal{A}\}$, then

$$\begin{tikzcd}
X \arrow[r, dashed, "p"] \arrow[d, dashed, "p_\alpha"'] \arrow[dr, dashed, "f"] & Y \arrow[d, dashed, "i"] \\
Z^{\alpha} \arrow[r, "q_\alpha"] & Z
\end{tikzcd}$$

$$p_\alpha = q_\alpha \cdot f \tag{21}$$

for a morphism $f$. Let

$$f = i \cdot p \tag{22}$$

the $(\mathcal{P}, \mathcal{I})$-factorization of the morphism $f$. Thus

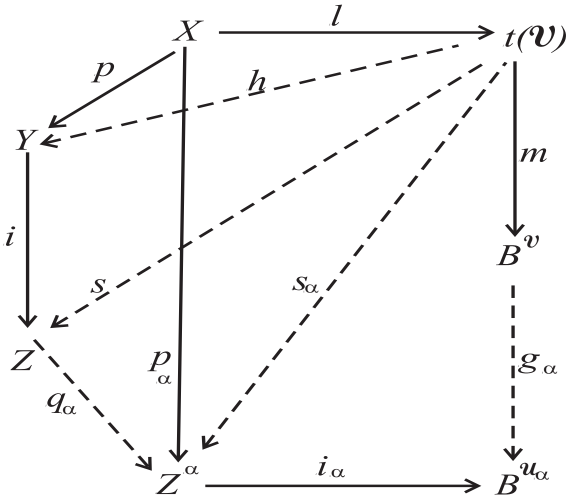
$$p \geq p_\alpha (\forall \alpha)$$

If $\mathcal{U} = \mathcal{E}xt_B(p)$, then $\mathcal{U}_\alpha \subset \mathcal{U}$ $(\forall \alpha)$. Let us verify that $\mathcal{U} = sup\{\mathcal{U}_\alpha | \alpha \in \mathcal{A}\}$. Let $\mathcal{V} \in S(X)$ and $\mathcal{U}_\alpha \subset \mathcal{V}$ $(\forall \alpha)$ be. Furthermore, let

$$m \cdot l = w^{\mathcal{V}}$$

the $(\mathcal{P}, \mathcal{I})$-factorization of respective morphisms. Because $\mathcal{U}_\alpha \subset \mathcal{V}$

$$i_\alpha \cdot p_\alpha = g_\alpha \cdot m \cdot l$$




for a morphism $g_\alpha$, with $l \in \mathcal{P}$ and $i_\alpha \in \mathcal{I}$. Then

$$p_\alpha = s_\alpha \cdot l, \tag{23}$$

$$g_\alpha \cdot m = i_\alpha \cdot s_\alpha (\forall \alpha)$$

for a morphism $s_\alpha$. In turn

$$s_\alpha = q_\alpha \cdot s(\forall \alpha). \tag{24}$$

We have

$$q_\alpha \cdot s \cdot l = (from(24)) = s_\alpha \cdot l = (from(23)) = p_\alpha = (from(21)) = q_\alpha \cdot f =$$

$$= (from(22)) = q_\alpha \cdot i \cdot p,$$

i.e.

$$q_\alpha \cdot s \cdot l = q_\alpha \cdot i \cdot p, (\forall \alpha)$$

or

$$s \cdot l = i \cdot p$$

with $l \in \mathcal{P}$ and $i \in \mathcal{I}$ $(l \perp i)$. Therefore

$$p = h \cdot l, \tag{25}$$

$$s = i \cdot h$$

for a morphism $h$.

From Theorem 3.4 it follows that $\mathcal{V} = \mathcal{E}xt_B(l)$, and similarly $\mathcal{U} = \mathcal{E}xt_B(p)$, and from the equality (25) we have $\mathcal{U} \subset \mathcal{V}$.

2. Let $f : X \to Y \in \mathcal{C}$ and $\mathcal{U} \in S(X)$.

On the morphisms $f$ and $p^{\mathcal{U}}$ we will construct the pullback square

$$p' \cdot f = f' \cdot p^{\mathcal{U}}.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p^{\mathcal{U}}"'] & Y \arrow[d, "p'"] & \\ t(\mathcal{U}) \arrow[r, "f'"] & T \arrow[r, "r^T"] & rT \end{tikzcd}$$

By definition we establish

$$S(f)(\mathcal{U}) = \mathcal{E}xt_B(r^T \cdot p').$$

By virtue of Theorem 3.4 $\mathcal{E}xt_B(r^T \cdot p') \in S(Y)$. ∎

**Theorem.** *Let the category $\mathcal{C}$ have the property PD. Then the map $S : \mathcal{C} \to \mathcal{L}at$ can be defined on the morphisms of the category $\mathcal{C}$ such that $S$ becomes a covariant functor $S : \mathcal{C} \to \mathcal{L}at$.*

*Proof.* Let

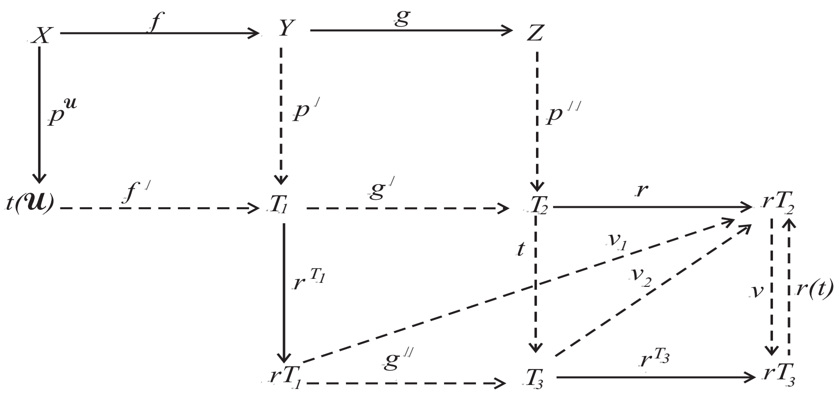
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

It is sufficient to verify that $S(g \cdot f) = S(g) \cdot S(f)$. Thus let $\mathcal{U} \in S(X)$

$$p' \cdot f = f' \cdot p^{\mathcal{U}},$$

$$p'' \cdot g = g' \cdot p'$$

the pullback squares built on morphisms $f$, $p^{\mathcal{U}}$ and $g$, $p'$.




Further, $r^{T_1} : T_1 \to rT_1$, $r^{T_2} : T_2 \to rT_2$ and $r^{T_3} : T_3 \to rT_3$ are the $\mathcal{R}$-replicas of the respective objects, where

$$g'' \cdot r^{T_1} = t \cdot g' \tag{26}$$

is the pullback square built on $r^{T_1}$ and $g'$. Thus

$$r^{T_3} \cdot t = r(t) \cdot r^{T_2}. \tag{27}$$

The morphism $r^{T_2} \cdot g'$ extends through the $\mathcal{R}$-replica of the object $T_1$:

$$r^{T_2} \cdot g' = v_1 \cdot r^{T_2}.$$

Since (26) is a pullback square, we have

$$r^{T_2} = v_2 \cdot t, \tag{28}$$

$$v_1 = v_2 \cdot g''$$

for a morphism $v_2$. Then the morphism $v_2$ extends through $r^{T_3}$:

$$v_2 = v \cdot r^{T_3}. \tag{29}$$

We have

$$v \cdot r(t) \cdot r^{T_2} = (from (27)) = v \cdot r^{T_3} \cdot t = (from (29)) = v_2 \cdot t = (from (28)) = r^{T_2},$$

i.e.

$$v \cdot r(t) \cdot r^{T_2} = r^{T_2},$$

or,

$$v \cdot r(t) = 1. \tag{30}$$

Also

$$r(t) \cdot v \cdot r^{T_3} \cdot t = (from (29)) = r(t) \cdot v_2 \cdot t = (from (28)) = r(t) \cdot r^{T_2} =$$

$$= (from(27)) = r^{T_3} \cdot t,$$

i.e.

$$r(t) \cdot v \cdot r^{T_3} \cdot t = r^{T_3} \cdot t.$$

In the pullback square (26), $r^{T_1} \in \mathcal{P}$, so $t \in \mathcal{P}$. Thus $r^{T_3} \cdot t \in \mathcal{P}$, and

$$r(t) \cdot v = 1. \tag{31}$$

From the formulas (30) and (31) it follows that

$$r(t) = v^{-1}.$$

Finally

$$S(g \cdot f)(\mathcal{U}) = \mathcal{E}xt_B(r^{T_2} \cdot p''),$$

since

$$(g' \cdot f') \cdot p^{\mathcal{U}} = p'' \cdot (g \cdot f)$$

is the pullback square built on $p^{\mathcal{U}}$, $g \cdot f$.

On the other hand,

$S(g)(S(f)(\mathcal{U})) = S(g)(\mathcal{E}xt_B(r^{T_1} \cdot p')) =$ (because $p^{\mathcal{E}xt_B(r^{T_1} \cdot p')} = r^{T_1} \cdot p'$ by virtue of Theorem 3.4.) $= \mathcal{E}xt_B(r^{T_3} \cdot t \cdot p'') = \mathcal{E}xt_B(r^{T_2} \cdot p'')$.

Thus we have proved that

$$S(g \cdot f) = S(g) \cdot S(f)). \blacksquare$$

## 4. THE CONTRAVARIANT FUNCTOR $\mathcal{S}^C : \mathcal{C} \to \mathcal{L}at$.

**4.1.** On objects, this functor operates in the same way as the functor $\mathcal{S} : \mathcal{C} \to \mathcal{L}at$:

$$\mathcal{S}^c(X) = \{\mathcal{U} \subset C(X, B) | \mathcal{U} \, \text{is saturated}\}.$$

Let $f : X \to Y \in \mathcal{C}$, $\mathcal{U} \in \mathcal{S}^c(Y)$ and

$$w^{\mathcal{U}} = i^{\mathcal{U}} \cdot p^{\mathcal{U}}$$

$(\mathcal{P}, \mathcal{I})$-factorization of the canonical morphism $w^{\mathcal{U}} : Y \to B^{\mathcal{U}}$. Further, let

$$p^{\mathcal{U}} \cdot f = i \cdot p \tag{32}$$

$(\mathcal{P}, \mathcal{I})$-factorization of the morphism $p^{\mathcal{U}} \cdot f$, and

$$\mathcal{V} = \mathcal{E}xt_B(p).$$

Also, let be

$$w^{\mathcal{U}} = i^{\mathcal{V}} \cdot p^{\mathcal{V}}$$

$(\mathcal{P}, \mathcal{I})$-factorization of the morphism $w^{\mathcal{V}} : X \to B^{\mathcal{V}}$.
We define

$$\mathcal{S}^c(f)(\mathcal{U}) = \mathcal{E}xt_B(p).$$

**Theorem.** *With the above notations the $\mathcal{P}$-factorobjects $p$ and $p\mathcal{V}$ are isomorphic: $p \sim p^{\mathcal{V}}$, and $\mathcal{V} \in \mathcal{S}(X)$. In particular $\mathcal{S}^c(f)(\mathcal{U}) \in \mathcal{S}^c(X)$.* ·

*Proof.* For any canonical projection $q_\alpha : B^{\mathcal{U}} \to B$ the morphism $q_\alpha \cdot i^{\mathcal{U}} \cdot i \cdot p \in \mathcal{V}$.
Thus

$$q_\alpha \cdot w^{\mathcal{U}} \cdot f = t_\alpha \cdot w^{\mathcal{V}}, (\forall \alpha) \tag{33}$$

for a morphism $t_\alpha : B^{\mathcal{V}} \to B$, and

$$t_\alpha = q_\alpha \cdot g_1, \tag{34}$$

for a morphism $g_1 : B^{\mathcal{V}} \to B^{\mathcal{U}}$. We have

$$q_\alpha \cdot w^{\mathcal{U}} \cdot f = (from(33)) = t_\alpha \cdot w^{\mathcal{V}} = (from(34)) = q_\alpha \cdot g_1 \cdot w^{\mathcal{V}}$$

i.e.

$$q_\alpha \cdot w^{\mathcal{U}} \cdot f = q_\alpha \cdot g_1 \cdot w^{\mathcal{V}}$$

or

$$w^{\mathcal{U}} \cdot f = g_1 \cdot w^{\mathcal{V}},$$

$$g_1 \cdot i^{\mathcal{V}} \cdot p^{\mathcal{V}} = i^{\mathcal{U}} \cdot p^{\mathcal{U}} \cdot f,$$

with $p^{\mathcal{V}} \in \mathcal{P}$ and $i^{\mathcal{U}} \in \mathcal{I}$ $(p^{\mathcal{U}} \perp i^{\mathcal{U}})$. Thus we have the diagonal $g_2 : t(\mathcal{V}) \to t(\mathcal{U})$ with the properties:

$$g_1 \cdot i^{\mathcal{V}} = i^{\mathcal{U}} \cdot g_2,$$

$$p^{\mathcal{U}} \cdot f = g_2 \cdot p^{\mathcal{V}}. \tag{35}$$

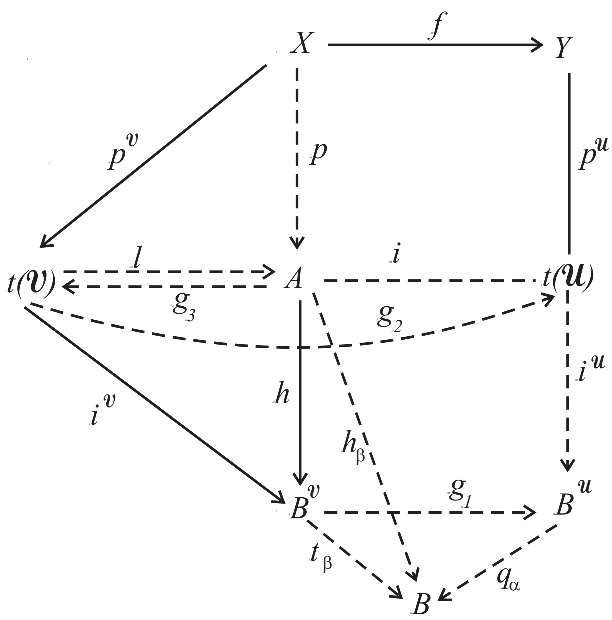
From (35) and (32) we have

$$i \cdot p = g_2 \cdot p^{\mathcal{V}},$$

with $p^{\mathcal{V}} \in \mathcal{P}$ and $i \in \mathcal{I}$ $(p^{\mathcal{U}} \perp i)$.
There exists diagonal $g_2 : t(\mathcal{V}) \to A$ with properties

$$g_2 = i \cdot g_3,$$

$$p = g_3 \cdot p^{\mathcal{V}}. \tag{36}$$




Further, for a canonical projection $t_\beta : B^{\mathcal{V}} \to B$ the morphism $t_\beta \cdot w^{\mathcal{V}} \in \mathcal{V}$, i.e. $t_\beta \cdot w^{\mathcal{V}}$ extends through $p$:

$$t_\beta \cdot w^{\mathcal{V}} = h_\beta \cdot p \tag{37}$$

with $h_\beta : A \to B$. Then

$$h_\beta = t_\beta \cdot h \tag{38}$$

for a $h : A \to B^{\mathcal{V}}$. We have

$$t_\beta \cdot i^{\mathcal{V}} \cdot p^{\mathcal{V}} = t_\beta \cdot w^{\mathcal{V}} = (from(37)) = h_\beta \cdot p = (from(38)) = t_\beta \cdot h \cdot p$$

i.e.

$$t_\beta \cdot i^{\mathcal{V}} \cdot p^{\mathcal{V}} = t_\beta \cdot h \cdot p$$

or

$$i^{\mathcal{V}} \cdot p^{\mathcal{V}} = h \cdot p \tag{39}$$

with $p \in \mathcal{P}$ and $i^{\mathcal{V}} \in \mathcal{I}$ $(p \perp i^{\mathcal{V}})$. We have diagonal $l : A \to t(\mathcal{V})$

$$p^{\mathcal{V}} = l \cdot p,$$

$$i^{\mathcal{V}} \cdot l = h. \tag{40}$$

Thus

$$i^{\mathcal{V}} \cdot l \cdot g_3 \cdot p^{\mathcal{V}} = (from(40)) = h \cdot g_3 \cdot p^{\mathcal{V}} = (from(36)) = h \cdot p = (from(39)) = i^{\mathcal{V}} \cdot p^{\mathcal{V}},$$

i.e.

$$i^{\mathcal{V}} \cdot l \cdot g_3 \cdot p^{\mathcal{V}} = i^{\mathcal{V}} \cdot p^{\mathcal{V}},$$

or

$$l \cdot g_3 = 1.$$

From (36) we deduce that $g_3 \in \mathcal{P}$, therefore $g_3 = h^{-1}$. Thus we have demonstrated that $\mathcal{V} = \mathcal{E}xt_B(p^{\mathcal{V}})$ and

$$\mathcal{S}^c(f)(\mathcal{U}) \in \mathcal{S}^c(X).$$

Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$\mathcal{U} \in \mathcal{S}^c(Z)$,

$$w^{\mathcal{U}} = i^{\mathcal{U}} \cdot p^{\mathcal{U}},$$

$$p^{\mathcal{U}} \cdot g = i_1 \cdot p_1,$$

$$p_1 \cdot f = i_2 \cdot p_2,$$

the $(\mathcal{P}, \mathcal{I})$-factorization of respective morphisms. Then

$$p^{\mathcal{U}} \cdot (g \cdot f) = (i_1 \cdot i_2) \cdot p_2,$$

is the $(\mathcal{P}, \mathcal{I})$-factorization of the morphism $p^{\mathcal{U}} \cdot (g \cdot f)$. Thus

$$\mathcal{S}^c(g \cdot f)(\mathcal{U}) = \mathcal{E}xt_B(p_2)$$

Further

$$\mathcal{S}^c(f)(\mathcal{S}^c(g)(\mathcal{U})) = \mathcal{S}^c(f)(\mathcal{E}xt_B(p_1)) = \mathcal{E}xt_B(p_2).$$

$$\mathcal{S}^c(g \cdot f)(\mathcal{U}) = \mathcal{S}^c(f)(\mathcal{S}^c(g)(\mathcal{U}))$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "p_2"] & Y \arrow[r, "g"] \arrow[d, dashed, "p_1"] & Z \arrow[d, "p^{u}"] \\
A_2 \arrow[r, dashed, "i_2"] & A_1 \arrow[r, dashed, "i_1"] & A
\end{tikzcd}$$

or

$$\mathcal{S}^c(g \cdot f) = \mathcal{S}^c(f) \cdot \mathcal{S}^c(g) \quad \blacksquare$$

# 5. THE FUNCTORIAL MORPHISMS $\boldsymbol{\alpha : \mathcal{S} \to \mathcal{F}}$ AND $\boldsymbol{\beta : \mathcal{F} \to \mathcal{S}}$.

## 5.1. The functorial morphisms $\alpha : \mathcal{S} \to \mathcal{F}$

As in paragraph 3, we will assume that $B$ is an $\mathcal{I}$-cogenerator of the subcategory $\mathcal{R}$: $\mathcal{R} \subset \mathcal{S}_{\mathcal{I}}\{B^{\tau} | \tau - cardinal\}$. Let $X \in |\mathcal{C}|$ and $\mathcal{U} \in \mathcal{S}(X)$. We define:

$$\alpha^X(\mathcal{U}) = p^{\mathcal{U}}, \tag{41}$$

where

$$w^{\mathcal{U}} = i^{\mathcal{U}} \cdot p^{\mathcal{U}}$$

is the $(\mathcal{P}, \mathcal{I})$-factorization of the canonical morphism

$$w^{\mathcal{U}} : X \to B^{C(X,B)}.$$

Let us verify that $\alpha$ is a functorial morphism. Let $f : X \to Y \in \mathcal{C}$, $\mathcal{U} \in \mathcal{S}(X)$ and

$$p' \cdot f = f' \cdot p^{\mathcal{U}}$$

the pullback squares built on morphisms $p^{\mathcal{U}}$ and $f$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "p^{\mathcal{U}}"'] & Y \arrow[d, dashed, "p'"] & \\
t(\mathcal{U}) \arrow[r, dashed, "f"] & Z \arrow[r, "r^Z"] & rZ
\end{tikzcd}$$

Then

$$\mathcal{S}(f)(\mathcal{U}) = \mathcal{E}xt_B(r^Z \cdot p') \tag{42}$$

and

$$\mathcal{F}(f)(p^{\mathcal{U}}) = r^Z \cdot p'. \tag{43}$$

Thus

$$\mathcal{F}(f)(\alpha^X(\mathcal{U})) = (from(41)) = \mathcal{F}(f)(p^{\mathcal{U}}) = (from(43)) = r^Z \cdot p'$$

i.e.

$$\mathcal{F}(f)(\alpha^X(\mathcal{U})) = r^Z \cdot p'. \tag{44}$$

Also

$\alpha^Y(\mathcal{S}(f)(\mathcal{U})) = (from(42)) = \alpha^Y(\mathcal{E}xt_B(r^Z \cdot p')) = (from(41)) = p^{\mathcal{E}xt_B(r^Z \cdot p')}$ $=$ (according to Theorem 3.4.) $= r^Z \cdot p'$, i.e.

$$\alpha^Y(\mathcal{S}(f)(\mathcal{U})) = r^Z \cdot p'. \tag{45}$$

From the (44) and (45) formulas it follows that

$$\mathcal{F}(f)(\alpha^X(\mathcal{U})) = \alpha^Y(\mathcal{S}(f)(\mathcal{U})),$$

or

$$\mathcal{F}(f) \cdot \alpha^X = \alpha^Y \cdot \mathcal{S}(f).$$

$$\begin{tikzcd}
\mathcal{S}(X) \arrow[r, "\mathcal{S}(f)"] \arrow[d, dashed, "\alpha^X"'] & \mathcal{S}(Y) \arrow[d, dashed, "\alpha^Y"] \\
\mathcal{F}(X) \arrow[r, dashed, "\mathcal{F}(f)"] & \mathcal{F}(Y)
\end{tikzcd}$$

**5.2. The functorial morphisms** $\beta : \mathcal{F} \to \mathcal{S}$.

Consider $X \in |\mathcal{C}|$, $p : X \to A \in \mathcal{F}(X)$ and $f : X \to Y \in \mathcal{C}\cdot$.
We note

$$\beta^X(p) = \mathcal{E}xt_B(p). \tag{46}$$

We have

$$\mathcal{F}(f)(p) = r^Z \cdot p', \tag{47}$$

where

$$p' \cdot f = f' \cdot p,$$

is the pullback squares built on morphisms $p$ and $f$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, dashed, "p'"] & \\
A \arrow[r, dashed, "f'"] & Z \arrow[r, "r^Z"] & rZ
\end{tikzcd}$$

We have

$$\beta^Y(\mathcal{F}(f)(p)) = (from(47)) = \beta^Y(r^Z \cdot p') = (from(46)) = \mathcal{E}xt_B(r^Z \cdot p')$$

i.e.

$$\beta^Y(\mathcal{F}(f)(p)) = \mathcal{E}xt_B(r^Z \cdot p'). \tag{48}$$

Also
$\mathcal{S}(f)(\beta^X(p)) = (from(46)) = \mathcal{S}(f)(\mathcal{E}xt_B(p)) =$ (according to Theorem 3.4) $= \mathcal{E}xt_B(r^Z \cdot p')$, i.e.

$$\mathcal{S}(f)(\beta^X(p)) = \mathcal{E}xt_B(r^Z \cdot p') \tag{49}$$

where

$$f' \cdot p^{\mathcal{U}} = p' \cdot f.$$

Because

$$\mathcal{U} = \mathcal{E}xt_B(p)$$

and

$$p^{\mathcal{U}} \sim p.$$

Thus, by virtue of formulas (48) and (49), we have:

$$\beta^Y(\mathcal{F}(f)(p)) = \mathcal{S}(f)(\beta^X(p))$$

or

$$\beta^Y \cdot \mathcal{F}(f) = \mathcal{S}(f) \cdot \beta^X.$$

**5.3. Theorem.** *Let $B$ be an $\mathcal{I}$-cogenerator of the subcategory $\mathcal{R}$. Then the covariant functors $\mathcal{F} : \mathcal{C} \to \mathcal{L}at$ and $\mathcal{S} : \mathcal{C} \to \mathcal{L}at$ are isomorphic, the isomorphism being given by: $\alpha : \mathcal{F} \to \mathcal{S}$, $\beta : \mathcal{S} \to \mathcal{F}$.*

*Proof.* $\beta \cdot \alpha = 1$. Consider $X \in |\mathcal{C}|$ and $\mathcal{U} \in \mathcal{S}(X)$. Then

$$\beta^X(\alpha^X(\mathcal{U})) = (from(41)p.5.1.) = \beta^X(p^{\mathcal{U}}) = (from(46)p.5.2.) = \mathcal{E}xt_B(p^{\mathcal{U}}) = \mathcal{U}.$$

The last equality is written by virtue of the definition of the saturated set (Definition 3.1).

$\alpha \cdot \beta = 1$. Let $X \in |\mathcal{C}|$ and $q : X \to A \in \mathcal{F}(X)$ be.

Then by Theorem 3.1. we have:

$\alpha^X(\beta^X(q)) = (from(46)p.5.2.) = \alpha^X(\mathcal{E}xt_B(q)) = (from(41)p.5.1.) = p^{\mathcal{E}xt_B(q)} = q.$ ∎

## 6. THE FUNCTORIAL MORPHISMS $\overline{\alpha} : \mathcal{S}^C \to \mathcal{F}^C$ AND $\overline{\beta} : \mathcal{F}^C \to \mathcal{S}^C$.

### 6.1. The functorial morphism $\overline{\alpha} : \mathcal{S}^c \to \mathcal{F}^c$.

Consider $X \in |\mathcal{C}|$ and $\mathcal{U} \in \mathcal{S}^c(X)$. We note:

$$\overline{\alpha}^X(\mathcal{U}) = p^{\mathcal{U}}, \tag{50}$$

where

$$w^{\mathcal{U}} = i^{\mathcal{U}} \cdot p^{\mathcal{U}}$$

is the $(\mathcal{P},\mathcal{I})$-factorization of the canonical morphism

$$w^{\mathcal{U}} : X \to B^{\mathcal{C}(X,B)}.$$

Let us verify that $\overline{\alpha}$ is a functorial morphism. Let be $f : X \to Y \in \mathcal{C}$, $\mathcal{U} \in \mathcal{S}^c(Y)$,

$$p^{\mathcal{U}} \cdot f = i \cdot p$$

the $\mathcal{P},\mathcal{I}$-factorization of respective morphism and

$$\mathcal{V} = \mathcal{E}xt_B(p).$$

By Theorem 4.1. $p \sim p^{\mathcal{V}}$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p^{\mathcal{V}}"'] & Y \arrow[d, "p^{\mathcal{U}}"] \\ t(\mathcal{V}) \arrow[r, "i"] & t(\mathcal{U}) \end{tikzcd}$$

We have:

$$\overline{\alpha}^X(\mathcal{S}^c(f)(\mathcal{U})) = p^{\mathcal{V}} = \overline{\alpha}^X(\mathcal{V}) = p,$$

Also, by p.2.1,

$$\mathcal{F}^c(f)(\overline{\alpha}^Y(\mathcal{U})) = \mathcal{F}^c(f)(p^{\mathcal{U}}) = p.$$

Thus, we have demonstrated that

$$\overline{\alpha}^X(\mathcal{S}^c(f) = \mathcal{F}^c(f)(\overline{\alpha}^Y).$$

$$\begin{tikzcd} \mathcal{S}^c(Y) \arrow[r, "\mathcal{S}^c(f)"] \arrow[d, "\bar{\alpha}^Y"'] & \mathcal{S}^c(X) \arrow[d, "\bar{\alpha}^X"] \\ \mathcal{F}^c(Y) \arrow[r, "\mathcal{F}^c(f)"] & \mathcal{F}^c(X) \end{tikzcd}$$

**6.2. The functorial morphisms $\overline{\beta} : \mathcal{F}^c \to \mathcal{S}^c$.**

For $q \in \mathcal{F}^c(X)$ we note

$$\overline{\beta}^X(q) = \mathcal{E}xt_B(q). \qquad (51)$$

Let $f : X \to Y$ and $q \in \mathcal{F}^c(Y)$ be.

We have $\overline{\beta}^X(\mathcal{F}^c(f)(q))$ = (see p.2.1)= $\overline{\beta}^X(p_1) = (from(1)) = \mathcal{E}xt_B(p_1)$, where

$$q \cdot f = i_1 \cdot p_1,$$

is the $(\mathcal{P}, \mathcal{I})$-factorization of the morphism $q \cdot f$ (see 4.1).

$$\begin{tikzcd}
\mathcal{F}^c(Y) \arrow[r, "\mathcal{F}^c(f)"] \arrow[d, "\bar{\beta}^Y"'] & \mathcal{F}^c(X) \arrow[d, "\bar{\beta}^X"] \\
\mathcal{S}^c(Y) \arrow[r, "\mathcal{S}^c(f)"] & \mathcal{S}^c(X)
\end{tikzcd}$$

Also $\mathcal{S}^c(f)(\overline{\beta}^X(q)) = (from(51)) = \mathcal{S}^c(f)(\mathcal{E}xt_B(q))$ = (because $p^{\mathcal{E}xt_B(q)} \sim q$) $= \mathcal{E}xt_B(p_1)$.

**6.3. Theorem.** *Let $B$ be an $\mathcal{I}$-cogenerator of the subcategory $\mathcal{R}$. Then the functorial morphisms $\overline{\alpha} : \mathcal{S}^c \to \mathcal{F}^c$ and $\overline{\beta} : \mathcal{F}^c \to \mathcal{S}^c$ establish a functorial isomorphism.*

*Proof.* $\overline{\beta} \cdot \overline{\alpha} = 1$. Let $X \in |\mathcal{C}|$ and $\mathcal{U} \in \mathcal{S}^c(X)$ be. We have

$$\overline{\beta}^X(\overline{\alpha}^X(\mathcal{U})) = (from(50)p.6.1.) = \overline{\beta}^X(p^{\mathcal{U}}) = (from(51)p.6.2.) = \mathcal{E}xt_B(p^{\mathcal{U}}) = \mathcal{U},$$

(see definitions 4.1 and 6.1.)

$\overline{\alpha}^X \cdot \overline{\beta}^X = 1$. Let $q \in \mathcal{F}^c(X)$ be.

Then:

$$\overline{\alpha}^X(\overline{\beta}^X(q)) = \overline{\alpha}^X(\mathcal{E}xt_B(q)) = (from(50)p.6.1.) = p^{\mathcal{U}},$$

where

$$w^{\mathcal{U}} = i^{\mathcal{U}} \cdot p^{\mathcal{U}}$$

is the $(\mathcal{P}, \mathcal{I})$-factorization of the canonical morphism $w^{\mathcal{U}}$. Since $B$ is an $\mathcal{I}$-cogenerator of the subcategory $\mathcal{R}$, then, by Theorem 3.4 $p^{\mathcal{U}} \sim q$. ∎

## 7. CONCLUSIONS.

The paper establishes a complete dictionary between two descriptions of the reflective quotients relative to the $\mathcal{P}$-reflective subcategory $\mathcal{R}$ with $\mathcal{I}$-cogenerator $B$. On the one hand, the lattice $\mathbb{P}(X) = \mathcal{F}(X)$ of $\mathcal{P}$-morphisms into $\mathcal{R}$; on the other, the lattice $\mathcal{S}(X)$ of saturated subsets of $\mathcal{C}(X, B)$.

The main theorems prove that: - both assignments are complete lattices under the hypotheses PS and PD (Lemmas 1.3, 1.5 and Theorem 3.5);

- both extend to covariant functors $\mathcal{F}$, $\mathcal{S} : \mathcal{C} \to \mathcal{L}at$ and to contravariant functors $\mathcal{F}^c$, $\mathcal{S}^c : \mathcal{C} \to \mathcal{L}at$ ;

- the natural transformations $\alpha$, $\beta$ (Theorem 5.3) and $\overline{\alpha}$, $\overline{\beta}$ (Theorem 6.3) are reciprocal inverse, so that $\mathcal{F} \approxeq \mathcal{S}$ and $\mathcal{F}^c \approxeq \mathcal{S}^c$ functorially. An essential role is played by the orthogonality $\mathcal{P} \perp \mathcal{I}$, which produces all diagonal morphisms in the proofs, and the cogenerating property of B, which guarantees that saturated families of maps into $B$ separate exactly the reflective quotients. As the examples show, the framework simultaneously covers the lattice of partitions of a set, Boolean algebras of clopen sets in zero-dimensional spaces and function-algebra descriptions of compactifications, suggesting further applications to other reflective subcategories of topological and algebraic categories.

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# INFINITELY MANY CONCRETE MULTIPLE-COMPOSITE QUANTITATIVE APPROXIMATIONS BY DIFFERENT MULTIVARIATE KANTOROVICH-CHOQUET NEURAL NETWORK OPERATORS

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**Abstract** Here we study the univariate and multivariate quantitative approximation by infinitely many concrete and different multi-composite Kantorovich-Choquet type quasi-interpolation neural network operators with respect to supremum norm. This is achieved with rates via the first univariate and multivariate moduli of continuity. We approximate continuous and bounded non-negative functions on $\mathbb{R}^N$, $N \in \mathbb{N}$. When they are also uniformly continuous we have pointwise and uniform convergences, plus $L_p$ estimates. Our multi-composite activation functions are formed by 7 specific known sigmoid functions.

**Keywords:** multi-composite specific sigmoid activation functions, quasi-interpolation neural network approximation, multi-composite Kantorovich-Choquet type operators.
**2020 MSC:** 41A17, 41A25, 41A30, 41A35.

## 1. INTRODUCTION

**Multi-composition of activation functions** refers to the technique of chaining, mixing, or stacking multiple activation functions, either in sequence or in parallel (concatenation) to create more complex, flexible, and often trainable non-linearities in neural networks. Instead of using a single activation (like ReLU) throughout the network, this approach leverages multiple functions to improve performance, enhance model expressiveness, and optimize learning.

Types of Activation Composition

- **Sequential Composition:** Applying one activation function after another (e.g., $f(g(x))$). A recent development includes "PolyCom" (Polynomial Composition), which combines polynomial functions with others like ReLU to accelerate convergence and improve accuracy. Another example is compounding functions to create "normalized cusp neural network operators", which can reduce infinite domains to compact supports, enhancing approximation capabilities.

- **Parallel Composition (Concatenation):** Computing the outputs of different activation functions $(f_1(x), f_2(x), ...)$ for the same input and concatenating them into a single vector. This allows the network to utilize multiple non-linearities at once.
- **Dynamic Activation Composition (Dyn):** A technique that uses learnable, normalized convex combinations of basis activation functions. This allows the network to adaptively "mix" activations during training, enhancing model adaptability and performance.
- **Multivariate/Multi-dimensional Activations:** Moving beyond simple scalar functions to functions that take multiple inputs and produce multiple outputs (e.g., generalizing ReLU to a Second-Order Cone projection). These approaches are shown to have higher expressive power than traditional single-input, single-output activations.

Advantages and Applications

- **Increased Expressiveness:** Complex compositions allow networks to learn more intricate patterns and handle non-linearly separable data more effectively.
- **Improved Accuracy:** Studies have shown that combining different activation functions can lead to better performance compared to using a single standard activation, particularly for less predictable data distributions.
- **Faster Convergence:** Certain compositions, such as multi-kernel activation functions (multi-KAF) or dynamic mixtures, can help models convergence faster by better adapting to the data.
- **Controlled Output Range:** Composing functions can limit the output to specific, desirable ranges, which helps in focusing on important information and filtering out noise.
- **Better Gradient Flow:** Some compositions, such as concatenating Swish and Tanh, can provide paths with non-zero derivatives, helping to mitigate vanishing gradient problems.

Key research Findings

- **Flexibility:** research indicates that compositing activation functions - such as using a "cusp" function (a composition of two functions) - results in more flexible and powerful neural networks.
- **Learned Activations:** Instead of manually selecting the composition, techniques like "Trainable Adaptive Activation Function Structure

(TAAFS)" learn the optimal combination of activation functions during training.

- **Efficiency Concerns:** While composing functions can improve accuracy, it may add to the computational load. However, the performance gains often justify the added complexity.

This approach is increasingly being explored to improve the performance of deep learning models, inclusing Transformers and Large Language Models (LLMs).

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [2] he gives the $N$th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

The author inspired by [8], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [2] - [6], [37], by treating both the univariate and multivariate cases. He did also the corresponding fractional case [3, 6].

The author here performs infinitely many multi-composite univariate and multivariate specific and concrete sigmoid activation functions relied neural network approximations to continuous non-negative functions over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$, then he extends his results to complex valued functions. $L_p$ approximations are also included. All convergences here are with rates expressed via the modulus of continuity of the involved function and given by very tight Jackson type inequalities.

The author comes up with the "right" precisely defined flexible model of quasi-interpolation, Kantorovich-Choquet type integral coefficient neural networks operators associated with: multi-composite 7 specific sigmoid activation functions. In preparation to prove our results we establish important properties of the multi-composite specific density functions defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this work, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of $a_j$ and $x$, and $\sigma$ is the activation function of the network. About neural networks in general read [10], [11], [14]-[16]. For recent related works see [17]-[36].

## 2. FOUNDATIONS

We need

**Definition 2.1.** ([6], p. 530) *The general sigmoid activation functions we deal with here are described as follows $h : \mathbb{R} \to [-1,1]$, strictly increasing $h(0) = 0$, $h(-x) = -h(x)$, $x \in \mathbb{R}$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also $h$ is striclty convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.*

More precisely we will deal with the following sigmoid activartion functions which possess all of the above properties: the algebraic activation function,

$$g_1(x) = \frac{x}{\sqrt[2m]{1+x^{2m}}}, \quad m \in \mathbb{N}, \, x \in \mathbb{R}, \tag{1}$$

(see [37], pp. 2-3);

the normalized and parametrized arctangent

$$g_2(x) = \frac{2}{\pi}\arctan\left(\frac{x}{2}\lambda x\right) = \frac{2}{\pi}\int_0^{\frac{\pi\lambda x}{2}} \frac{dz}{1+z^2}, \quad x \in \mathbb{R}, \, \lambda > 0, \tag{2}$$

(see [6], p.p. 156-157);

the normalized and parametrized Gudermannian function

$$g_3(x) = \frac{2}{\pi} gd(\lambda x) = \frac{4}{\pi}\arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right)$$

$$= \frac{2}{\pi}\int_0^{\lambda x} \frac{dt}{\cosh t} = \frac{4}{\pi}\int_0^{\lambda x} \frac{dt}{e^t + e^{-t}}, \quad x \in \mathbb{R}, \, \lambda > 0, \tag{3}$$

where

$$gd(x) := \int_0^x \frac{dt}{\cosh t} = 2\arctan\left(\tanh\left(\frac{x}{2}\right)\right), \quad \forall \, x \in \mathbb{R},$$

is the Gudermannian function (see [6], p. 198-199);

the parametrized error function

$$g_4(x) = erf \lambda x = \frac{2}{\sqrt{\pi}}\int_0^{\lambda x} e^{-t^2} dt, \quad x \in \mathbb{R}, \, \lambda > 0, \tag{4}$$

(see [6], pp. 226-227);

the parametrized hyperbolic tangent function

$$g_5(x) = \tanh \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}} = \frac{e^{2\lambda x} - 1}{e^{2\lambda x} + 1}$$

$$= \frac{1 - e^{-2\lambda x}}{1 + e^{-2\lambda x}}, \quad x > 0, \ \lambda > 0, \tag{5}$$

(see [6], p. 72);

for small $0 < \lambda < 1$, $\tanh \lambda x$ is expected to behave better than *ReLu* and *Leaky ReLu* activation functions;

the parametrized hyoerbolic tangent like activation function,

$$g_6(x) = \frac{A^{\lambda x} - A^{-\lambda x}}{A^{\lambda x} + A^{-\lambda x}}, \quad A > 1, \ \lambda > 0, \ x \in \mathbb{R} \tag{6}$$

(see [6], pp. 262-263);

finally, the $\beta$-parametrized half hyperbolic tangent function

$$g_7(x) = \frac{1 - e^{-\beta x}}{1 + e^{-\beta x}}, \quad \beta > 0, \ \forall \ x \in \mathbb{R}, \tag{7}$$

(see [6], p. 509-510).

So, we denote them by $g_i(x)$, $x \in \mathbb{R}$, $i = 1, 2, ..., 7$.

Here we deal with the specific sequences of activation functions $\left(h_i^{(r)}\right)$, $r \in \mathbb{N}$, $i \in \mathbb{N}$, such that all $h_i^{(r)} \in G := \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$. That is $h_i^{(r)}$ for different $i \in \mathbb{N}$ could be identical, $r \in \mathbb{N}$.

Therefore in the sequences of activation functions $h_1^{(r)}, h_2^{(r)}, ...$, we allow random repetitions. Thus any two random $h_i^{(r)}$ activarion functions are not necessarily different, but all belong to $G$. Call $H^{(r)} = \left\{h_1^{(r)}, h_2^{(r)}, ...\right\}$, $r \in \mathbb{N}$, and by $\Omega := \left\{H^{(r)}, \text{ all } r \in \mathbb{N}\right\}$. For simplicity next we can consider such a sequence $\{h_1, h_2, ...\}$.

Notice here $0 < h_i(1) \leq 1$, $i = 1, 2, ....$ Any composition $h_1 \circ h_2 \circ h_3 \circ ... \circ h_\lambda$ is meant to be $h_1|_{[-1,1]} \circ h_2|_{[-1,1]} \circ h_3|_{[-1,1]} \circ ... \circ h_{\lambda-1}|_{[-1,1]} \circ h_\lambda$, $\lambda \in \mathbb{N}$, and for convenience, we denote it by $G_\lambda := h_1 \circ h_2 \circ h_3 \circ ... \circ h_\lambda$. We have for any $\lambda \in \mathbb{N} : 0 < h_\lambda(1) \leq 1$, hence $0 < h_{\lambda-1}(h_\lambda(1)) \leq h_{\lambda-1}(1) \leq 1$, and $0 < h_{\lambda-2}(h_{\lambda-1}(h_\lambda(1))) \leq h_{\lambda-2}(h_{\lambda-1}(1)) \leq h_{\lambda-2}(1) \leq 1$.

Inductively we derive that $0 < G_\lambda(1) \leq 1$, $\forall \ \lambda \in \mathbb{N}$.

Clearly, it us $G_\lambda(0) = 0$ and $G_\lambda$ is strictly increasing over $\mathbb{R}$. Furthermore it holds

$$G_\lambda(-x) = h_1(h_2(h_3(...(h_{\lambda-1}(h_\lambda(-x)))))) = h_1(h_2(h_3(...(h_{\lambda-1}(-h_\lambda(x))))))$$

$$= ... = -h_1(h_2(h_3(...(h_{\lambda-1}(h_\lambda(x)))))) = -G_\lambda(x), \quad x \in \mathbb{R}.$$

Clearly it holds $G_{\lambda}^{(2)} \in C(\mathbb{R})$.

We notice that

$$G_{\lambda}(+\infty) = h_1(h_2(h_3(...(h_{\lambda-1}(h_{\lambda}(+\infty)))))) =$$
$$h_1(h_2(h_3(...(h_{\lambda-1}(1))))) = G_{\lambda-1}(1), \tag{8}$$

and

$$G_{\lambda}(-\infty) = h_1(h_2(h_3(...(h_{\lambda-1}(h_{\lambda}(-\infty)))))) = h_1(h_2(h_3(...(h_{\lambda-1}(-1)))))$$
$$= -h_1(h_2(h_3(...(h_{\lambda-1}(1))))) = -G_{\lambda-1}(1). \tag{9}$$

Consequently, it holds

$$-G_{\lambda-1}(1) \leq G_{\lambda}(x) \leq G_{\lambda-1}(1), \quad \forall\, x \in \mathbb{R}. \tag{10}$$

Thus, $y = \pm G_{\lambda-1}(1)$ are asymptotes of $G_{\lambda}(x)$, any $\lambda \in \mathbb{N}$.

Next we act over $(-\infty, 0]$ : let $\lambda, \mu \geq 0 : \lambda + \mu = 1$. Then by convexity of $h_2$ there we have

$$h_2(\lambda x + \mu y) \leq \lambda h_2(x) + \mu h_2(y), \quad \forall\, x, y \in (-\infty, 0];$$

and

$$h_1(h_2(\lambda x + \mu y)) \leq h_1(\lambda h_2(x) + \mu h_2(y)) \leq$$
$$\lambda(h_1 \circ h_2)(x) + \mu(h_1 \circ h_2)(y), \quad \forall x,\ y \in (-\infty, 0]. \tag{11}$$

So that $h_1 \circ h_2$ is convex over $(-\infty, 0]$.

Now we work on $[0, +\infty)$ : let $\lambda, \mu \geq 0 : \lambda + \mu = 1$. Then by concavity of $h_2$ there we have

$$h_2(\lambda x + \mu y) \geq \lambda h_2(x) + \mu h_2(y), \quad \forall\, x, y \in [0, +\infty);$$

and

$$h_1(h_2(\lambda x + \mu y)) \geq h_1(\lambda h_2(x) + \mu h_2(y)) \geq$$
$$\lambda(h_1 \circ h_2)(x) + \mu(h_1 \circ h_2)(y), \quad \forall x,\ y \in [0, +\infty). \tag{12}$$

Thus, $h_1 \circ h_2$ is concave over $[0, +\infty)$.

Therefore $G_2 = h_1 \circ h_2$ is a general sigmoid activation function with asymptotes $y = \pm h_1(1)$, and fulfilling the rest of conditions of Definition 2.1.

Arguing as above $h_2 \circ h_3 : \mathbb{R} \to [-1, 1]$, fulfills Definition 2.1, and $h_1 \circ h_2 \circ h_3$ does the same with asymptotes $h = \pm h_1(h_2(1))$.

Inductively, we prove that $G_{\lambda}$ fulfills Definition 2.1 with asymptotes $y = \pm G_{\lambda-1}(1)$.

We have established the following:

**Theorem 2.1.** *Let $r, \lambda \in \mathbb{N}$. Then $G_{\lambda}^{(r)} := h_1^{(r)} \circ h_2^{(r)} \circ h_3^{(r)} \circ ... \circ h_{\lambda}^{(r)}$ fulfill all the properties of Definition 2.1 with asymptotes $y = \pm G_{\lambda-1}^{(r)}(1)$. That is $G_{\lambda}^{(r)}$ are multi-composite general sigmoid activation functions from $\mathbb{R} \to [-1,1]$.*

**Corollary 2.1.** $\frac{G_{\lambda}^{(r)}}{G_{\lambda-1}^{(r)}(1)}$ *fulfill Definition 2.1 with asymptotes $y = \pm 1$.*

We call

$$\widetilde{G}_{\lambda}^{(r)} := \frac{G_{\lambda}^{(r)}}{G_{\lambda-1}^{(r)}(1)}. \tag{13}$$

**Remark 2.1.** *Next we consider the functions*

$$T_{\lambda}^{(r)}(x) := \frac{1}{4}\left(\widetilde{G}_{\lambda}^{(r)}(x+1) - \widetilde{G}_{\lambda}^{(r)}(x-1)\right) > 0, \quad \forall\, x \in \mathbb{R}, \; r, \lambda \in \mathbb{N}. \tag{14}$$

*We observe that*

$$\begin{gathered} T_{\lambda}^{(r)}(-x) = \frac{1}{4}\left(\widetilde{G}_{\lambda}^{(r)}(-x+1) - \widetilde{G}_{\lambda}^{(r)}(-x-1)\right) = \\ \frac{1}{4}\left(\widetilde{G}_{\lambda}^{(r)}(-(x-1)) - \widetilde{G}_{\lambda}^{(r)}(-(x+1))\right) = \frac{1}{4}\left(-\widetilde{G}_{\lambda}^{(r)}(x-1) + \widetilde{G}_{\lambda}^{(r)}(x+1)\right) = \\ \frac{1}{4}\left(\widetilde{G}_{\lambda}^{(r)}(x+1) - \widetilde{G}_{\lambda}^{(r)}(x-1)\right) = T_{\lambda}^{(r)}(x). \end{gathered} \tag{15}$$

*That is $T_{\lambda}^{(r)}$ are even functions,*

$$T_{\lambda}^{(r)}(-x) = T_{\lambda}^{(r)}(x), \quad \forall\, x \in \mathbb{R}, \; r, \lambda \in \mathbb{N}. \tag{16}$$

*We see that*

$$T_{\lambda}^{(r)}(0) = \frac{\widetilde{G}_{\lambda}^{(r)}(1)}{2}. \tag{17}$$

*Let $x > 1$, we have that*

$$T_{\lambda}^{(r)\prime}(x) = \frac{1}{4}\left(\widetilde{G}_{\lambda}^{(r)\prime}(x+1) - \widetilde{G}_{\lambda}^{(r)\prime}(x-1)\right) < 0,$$

*by $\widetilde{G}_{\lambda}^{(r)\prime}$ being strictly decreasing over $[0, +\infty)$.*

*Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $\widetilde{G}_{\lambda}^{(r)\prime}(x-1) = \widetilde{G}_{\lambda}^{(r)\prime}(1-x) > \widetilde{G}_{\lambda}^{(r)\prime}(x+1)$, so that again $T_{\lambda}^{(r)\prime}(x) < 0$. Consequently $T_{\lambda}^{(r)}$ is strictly decreasing on $(0, +\infty)$.*

*Clearly, $T_{\lambda}^{(r)}$ is strictly increasing on $(-\infty, 0)$, and $T_{\lambda}^{(r)\prime}(0) = 0$.*

*Observe that*

$$\lim_{x\to+\infty} T_{\lambda}^{(r)}(x)=\frac{1}{4}\left(\widetilde{G}_{\lambda}^{(r)}(+\infty)-\widetilde{G}_{\lambda}^{(r)}(+\infty)\right)=0, \tag{18}$$

*and*

$$\lim_{x\to-\infty} T_{\lambda}^{(r)}(x)=\frac{1}{4}\left(\widetilde{G}_{\lambda}^{(r)}(-\infty)-\widetilde{G}_{\lambda}^{(r)}(-\infty)\right)=0, \quad r,\lambda\in\mathbb{N}. \tag{19}$$

*That is the $x$-axis is the horizontal asymptote on $T_{\lambda}^{(r)}$.*

*As a result $T_{\lambda}^{(r)}$ is a bell shaped symmetric function with maximum*

$$T_{\lambda}^{(r)}(0)=\frac{\widetilde{G}_{\lambda}^{(r)}(1)}{2}. \tag{20}$$

We need

**Theorem 2.2.** *It holds*

$$\sum_{i=-\infty}^{\infty} T_{\lambda}^{(r)}(x-i)=1, \ \ \forall\, x\in\mathbb{R}; \quad r,\lambda\in\mathbb{N}. \tag{21}$$

*Proof.* As similar to [6], p. 286 is omitted. ∎

**Theorem 2.3.** *We have that*

$$\int_{-\infty}^{\infty} T_{\lambda}^{(r)}(x)\,dx=1, \quad r,\lambda\in\mathbb{N}. \tag{22}$$

*Proof.* As similar to [6], p. 287, it is omitted. ∎

So that $T_{\lambda}^{(r)}(x)$ can serve as a density function in general.

We need

**Theorem 2.4.** ([7]) *Let $0<\alpha<1$, and $n\in\mathbb{N}$ with $n^{1-\alpha}>2$. Then*

$$\sum_{\begin{cases} k=-\infty \\ :|nx-k|\geq n^{1-\alpha}\end{cases}}^{\infty} T_{\lambda}^{(r)}(nx-k)<\left(1-\widetilde{G}_{\lambda}^{(r)}\left(n^{1-\alpha}-2\right)\right), \tag{23}$$

*and*

$$\lim_{n\to+\infty}\left(1-\widetilde{G}_{\lambda}^{(r)}\left(n^{1-\alpha}-2\right)\right)=0. \tag{24}$$

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

We also need

**Theorem 2.5.** *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(r)}(nx-k)} < \frac{1}{T_{\lambda}^{(r)}(1)}, \quad \forall\, x \in [a,b]; \quad r, \lambda \in \mathbb{N}. \tag{25}$$

*Proof.* As similar to [6], p. 289 is omitted. ∎

**Remark 2.2.** *We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(r)}(nx-k) \neq 1; \quad r, \lambda \in \mathbb{N}, \tag{26}$$

*for at least some $x \in [a,b]$.*

*See [6], p. 290, same reasoning.*

For large enough $n$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds (by (21))

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(r)}(nx-k) \leq 1. \tag{27}$$

We make

**Remark 2.3.** *We define*

$$\widehat{Z}_{\lambda}^{(r)}(x_1,...,x_N) := \widehat{Z}_{\lambda}^{(r)}(x) := \prod_{i=1}^{N} T_{\lambda}^{(r)}(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N, \ r, \lambda, N \in \mathbb{N}. \tag{28}$$

*It has the properties:*

*(i)*

$$\widehat{Z}_{\lambda}^{(r)}(x) > 0, \ \forall\, x \in \mathbb{R}^N, \tag{29}$$

*(ii)*

$$\sum_{k=-\infty}^{\infty} \widehat{Z}_{\lambda}^{(r)}(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \widehat{Z}_{\lambda}^{(r)}(x_1-k_1,...,x_N-k_N) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \prod_{i=1}^{N} T_{\lambda}^{(r)}(x_i-k_i) = \prod_{i=1}^{N} \left( \sum_{k_i=-\infty}^{\infty} T_{\lambda}^{(r)}(x_i-k_i) \right) \overset{(21)}{=} 1.$$

*Hence*

$$\sum_{k=-\infty}^{\infty} \widehat{Z}_{\lambda}^{(r)}(x-k) = 1, \quad r, \lambda \in \mathbb{N}. \tag{30}$$

*That is*

*(iii)*

$$\sum_{k=-\infty}^{\infty} \widehat{Z}_{\lambda}^{(r)}(nx-k)=1, \quad \forall x \in \mathbb{R}^N; \ r,\lambda,n \in \mathbb{N}. \tag{31}$$

*And*

*(iv)*

$$\int_{\mathbb{R}^N} \widehat{Z}_{\lambda}^{(r)}(x)\,dx = \int_{\mathbb{R}^N}\left(\prod_{i=1}^{N} T_{\lambda}^{(r)}(x_i)\right)dx_1...dx_N = \prod_{i=1}^{N}\left(\int_{-\infty}^{\infty} T_{\lambda}^{(r)}(x_i)\,dx_i\right) \overset{(22)}{=} 1, \tag{32}$$

*thus*

$$\int_{\mathbb{R}^N} \widehat{Z}_{\lambda}^{(r)}(x)\,dx = 1, \tag{33}$$

*that is* $\widehat{Z}_{\lambda}^{(r)}$ *is a multivariate density function,* $r,\lambda \in \mathbb{N}$.

*Here denote* $x=(x_1,...,x_N)$, $\|x\|_{\infty} := \max\{|x_1|,...,|x_N|\}$, $x \in \mathbb{R}^N$, *also set* $\infty := (\infty,...,\infty)$, $-\infty := (-\infty,...,-\infty)$ *upon the multivariate context,* $0<\beta<1$,

*(v) We have*

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} \widehat{Z}_{\lambda}^{(r)}(nx-k) = \sum_{\substack{k_1=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} ... \sum_{k_N=-\infty}^{\infty}\left(\prod_{i=1}^{N} T_{\lambda}^{(r)}(nx_i-k_i)\right) =$$

$$\prod_{i=1}^{N}\left(\sum_{\begin{cases} k_i=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} T_{\lambda}^{(r)}(nx_i-k_i)\right) \leq \quad \textit{(for some } r \in \{1,...,N\}\textit{ )}$$

$$\left(\prod_{\substack{i=1 \\ i\neq r}}^{N}\left(\sum_{k_i=-\infty}^{\infty} T_{\lambda}^{(r)}(nx_i-k_i)\right)\right)\left(\sum_{\begin{cases} k_r=-\infty \\ \left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^{\beta}} \end{cases}}^{\infty} T_{\lambda}^{(r)}(nx_r-k_r)\right) = \tag{34}$$

$$\sum_{\begin{cases} k_r=-\infty \\ \left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^{\beta}} \end{cases}}^{\infty} T_{\lambda}^{(r)}(nx_r-k_r) = \sum_{\begin{cases} k_r=-\infty \\ |nx_r-k_r| > n^{1-\beta} \end{cases}}^{\infty} T_{\lambda}^{(r)}(nx_r-k_r) \overset{(23)}{<}$$

$$1 - \widetilde{G}_{\lambda}^{(r)} \left( n^{1-\beta} - 2 \right).$$

*That is*

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} \widehat{Z}_{\lambda}^{(r)} (nx - k) < 1 - \widetilde{G}_{\lambda}^{(r)} \left( n^{1-\beta} - 2 \right). \tag{35}$$

$0 < \beta < 1$, $r, \lambda, n \in \mathbb{N} : n^{1-\beta} > 2$, $\forall\ x \in \prod_{i=1}^{N} [a_i, b_i]$.

Denote by

$$ {}_{\lambda}\widehat{\delta}_{N}^{(r)} (\beta, n) := 1 - \widetilde{G}_{\lambda}^{(r)} \left( n^{1-\beta} - 2 \right), \ \ r, \lambda \in \mathbb{N}, \tag{36}$$

$0 < \beta < 1$.

For $f \in C_B^+ \left( \mathbb{R}^N \right)$ (continuous and bounded functions from $\mathbb{R}^N$ into $\mathbb{R}_+$), we define the first modulus of continuity

$$\omega_1 (f, \delta) := \sup_{\substack{x, y \in \mathbb{R}^N \\ \|x - y\|_{\infty} \leq h}} |f(x) - f(y)|, \ \ h > 0. \tag{37}$$

Given that $f \in C_U^+ \left( \mathbb{R}^N \right)$ (uniformly continuous from $\mathbb{R}^N$ into $\mathbb{R}_+$, same definition for $\omega_1$), we have that

$$\lim_{h \to 0} \omega_1 (f, h) = 0. \tag{38}$$

When $N = 1$, $\omega_1$ is defined as in (37) with $\|\cdot\|_{\infty}$ collapsing to $|\cdot|$ and has the property (38).

## 3. BACKGROUND

### 3.1. DESCRIPTION OF CHOQUET INTEGRAL ([9])

We set:

**Definition 3.1.** *Consider* $\Omega \neq \varnothing$ *and let* $\mathcal{C}$ *be a* $\sigma$*-algebra of subsets in* $\Omega$*.*

*(i) (see, e.g., [33], p. 63) The set function* $\mu : \mathcal{C} \to [0, +\infty]$ *is called a monotone set function (or capacity) if* $\mu(\varnothing) = 0$ *and* $\mu(A) \leq \mu(B)$ *for all* $A, B \in \mathcal{C}$*, with* $A \subset B$*. Also,* $\mu$ *is called submodular if*

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \textit{ for all } A, B \in \mathcal{C}.$$

$\mu$ *is called bounded if* $\mu(\Omega) < +\infty$ *and normalized if* $\mu(\Omega) = 1$*.*

*(ii) (see, e.g., [33], p. 233) If* $\mu$ *is a monotone set function on* $\mathcal{C}$ *and if* $f : \Omega \to \mathbb{R}$ *is* $\mathcal{C}$*-measurable (that is, for any Borel subset* $B \subset \mathbb{R}$ *it follows* $f^{-1}(B) \in \mathcal{C}$*), the for any* $A \in \mathcal{C}$*, the Choquet integral is defined by*

$$(C)\int_A f d\mu = \int_0^{+\infty} \mu(F_\beta(f) \cap A)\, d\beta + \int_{-\infty}^0 [\mu(F_\beta(f) \cap A) - \mu(A)]\, d\beta,$$

*where we used the notation* $F_\beta(f) = \{\omega \in \Omega : f(\omega) \geq \beta\}$*. Notice that if* $f \geq 0$ *on* $A$*, then in the above formula we get* $\int_{-\infty}^0 = 0$.

*The integrals on the right-hand side are the usual Riemann integral.*

*The function* $f$ *will be called Choquet integrable on* $A$ *if* $(C)\int_A f d\mu \in \mathbb{R}$.

Next we list some well known properties of the Choquet integral.

**Remark 3.1.** *If* $\mu : \mathcal{C} \to [0, +\infty]$ *is a monotone set function, then the following properties hold:*

*(i) For all* $a \geq 0$ *we have* $(C)\int_A af d\mu = a \cdot (C)\int_A f d\mu$ *(if* $f \geq 0$ *then see, e.g., [33], Theorem 11.2, (5), p. 228 and if* $f$ *is arbitrary sign, then see, e.g., [12], p. 64, Proposition 5.1, (ii)).*

*(ii) For all* $c \in \mathbb{R}$ *and* $f$ *of arbitrary sign, we have (see, e.g., [33], pp. 232-233, or [12], p. 65)* $(C)\int_A (f + c)\, d\mu = (C)\int_A f d\mu + c \cdot \mu(A)$.

*If* $\mu$ *is submodular too, then for all* $f, g$ *of arbitrary sign and lower bounded, we have (see, e.g., [12], p. 75, Theorem 6.3)*

$$(C)\int_A (f + g)\, d\mu \leq (C)\int_A f d\mu + (C)\int_A g d\mu.$$

*(iii) If* $f \leq g$ *on* $A$ *then* $(C)\int_A f d\mu \leq (C)\int_A g d\mu$ *(see, e.g., [33], p. 228, Theorem 11.2, (3) if* $f, g \geq 0$ *and p. 232 if* $f, g$ *are of arbitrary sign).*

*(iv) Let* $f \geq 0$*. If* $A \subset B$ *then* $(C)\int_A f d\mu \leq (C)\int_B f d\mu$*. In addition, if* $\mu$ *is finitely subadditive, then*

$$(C)\int_{A \cup B} f d\mu \leq (C)\int_A f d\mu + (C)\int_B f d\mu.$$

*(v) It is immediate that* $(C)\int_A 1 \cdot d\mu(t) = \mu(A)$.

*(vi) If* $\mu$ *is a countably additive bounded measure, then the Choquet integral* $(C)\int_A f d\mu$ *reduces to the usual Lebesgue type integral (see, e.g., [12], p. 62, or [33], p. 226).*

*(vii) If* $f \geq 0$*, then* $(C)\int_A f d\mu \geq 0$.

*(viii) If* $\Omega = \mathbb{R}^N$*,* $N \in \mathbb{N}$*, we call* $\mu$ *strictly positive if* $\mu(A) > 0$*, for any open subset* $A \subseteq \mathbb{R}^N$*. See also [13].*

## 4. MAIN RESULTS

We need

**Definition 4.1.** *Let $\mathcal{L}$ be the Lebesgue $\sigma$-algebra on $\mathbb{R}^N$, $r, \lambda, N \in \mathbb{N}$, and the set function $\mu : \mathcal{L} \to [0, +\infty)$, which is assumed to be monotone, submodular and strictly positive. For $f \in C_B^+\left(\mathbb{R}^N\right)$, we define the countably many specific multi-composite multivariate Kantorovich-Choquet type neural network operators for any $x \in \mathbb{R}^N$:*

$$ {}_{\lambda}^{(r)}\widehat{K}_n^{\mu}(f, x) = {}_{\lambda}^{(r)}\widehat{K}_n^{\mu}(f, x_1, ..., x_N) := \tag{39}$$

$$\sum_{k=-\infty}^{\infty} \left( \frac{(C)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0, \frac{1}{n}\right]^N\right)} \right) \widehat{Z}_{\lambda}^{(r)}(nx - k) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \left( \frac{(C)\int_0^{\frac{1}{n}} \ldots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, ..., t_N + \frac{k_N}{n}\right) d\mu(t_1, ..., t_N)}{\mu\left(\left[0, \frac{1}{n}\right]^N\right)} \right)$$

$$\left( \prod_{i=1}^{N} T_{\lambda}^{(r)}(nx_i - k_i) \right),$$

*where $x = (x_1, ..., x_N) \in \mathbb{R}^N$, $k = (k_1, ..., k_N)$, $t = (t_1, ..., t_N)$, $n \in \mathbb{N}$.*

*Clearly here $\mu\left(\left[0, \frac{1}{n}\right]^N\right) > 0$, $\forall\ n \in \mathbb{N}$.*

*Above we notice that*

$$\left\| {}_{\lambda}^{(r)}\widehat{K}_n^{\mu}(f) \right\|_{\infty} \le \|f\|_{\infty}, \tag{40}$$

*so that ${}_{\lambda}^{(r)}\widehat{K}_n^{\mu}(f, x)$ is well-defined.*

We make

**Remark 4.1.** *Let $f \in C_B^+\left(\mathbb{R}^N\right)$, $t \in \left[0, \frac{1}{n}\right]^N$ and $x \in \mathbb{R}^N$, then*

$$f\left(t + \frac{k}{n}\right) = f\left(t + \frac{k}{n}\right) - f(x) + f(x) \le \left| f\left(t + \frac{k}{n}\right) - f(x) \right| + f(x),$$

*hence*

$$(C)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \frac{k}{n}\right) d\mu(t) \le$$

$$(C)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| d\mu(t) + (C)\int_{\left[0,\frac{1}{n}\right]^N} f(x)\, d\mu(t) = \tag{41}$$

$$(C)\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\frac{k}{n}\right)-f(x)\right|d\mu(t)+f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right).$$

*That is*

$$(C)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right)d\mu(t)-f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right)\leq$$

$$(C)\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\frac{k}{n}\right)-f(x)\right|d\mu(t). \tag{42}$$

*Similarly, we have that*

$$f(x)=f(x)-f\left(t+\frac{k}{n}\right)+f\left(t+\frac{k}{n}\right)\leq\left|f\left(t+\frac{k}{n}\right)-f(x)\right|+f\left(t+\frac{k}{n}\right).$$

*Hence*

$$(C)\int_{\left[0,\frac{1}{n}\right]^N} f(x)\,\mu(dt)\leq$$

$$(C)\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\frac{k}{n}\right)-f(x)\right|d\mu(t)+(C)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right)\mu(dt),$$

*and*

$$f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right)-(C)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right)\mu(dt)\leq$$

$$(C)\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\frac{k}{n}\right)-f(x)\right|d\mu(t). \tag{43}$$

*By (42) and (43) we derive that*

$$\left|(C)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right)\mu(dt)-f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right)\right|\leq$$

$$(C)\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\frac{k}{n}\right)-f(x)\right|d\mu(t). \tag{44}$$

*In particular, it holds*

$$\left|\left(\frac{(C)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right)\mu(dt)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}\right)-f(x)\right|\leq\frac{(C)\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\frac{k}{n}\right)-f(x)\right|d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}. \tag{45}$$

We give the following approximation result.

**Theorem 4.1.** *Let* $f\in C_B^+\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x\in\mathbb{R}^N$, $\lambda,r,N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$. *Then* *i)*

$$\sup_{\mu}\left|{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f,x\right)-f\left(x\right)\right|\leq\omega_{1}\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+2\left\|f\right\|_{\infty}\ {}_{\lambda}\widehat{\delta}_{N}^{(r)}\left(\beta,n\right)=:\ {}_{\lambda}\sigma_{n}^{(r)}, \tag{46}$$

*and*

*ii)*

$$\sup_{\mu}\left\|{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f\right)-f\right\|_{\infty}\leq\ {}_{\lambda}\sigma_{n}^{(r)}. \tag{47}$$

*Given that* $f\in\left(C_{U}^{+}\left(\mathbb{R}^{N}\right)\cap C_{B}^{+}\left(\mathbb{R}^{N}\right)\right)$, *we obtain* $\lim_{n\to\infty}{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f\right)=f$, *uniformly.*

*Proof.* We observe that

$$\left|{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f,x\right)-f\left(x\right)\right|=$$

$$\left|\sum_{k=-\infty}^{\infty}\left(\frac{(C)\int_{\left[0,\frac{1}{n}\right]^{N}}f\left(t+\frac{k}{n}\right)d\mu\left(t\right)}{\mu\left(\left[0,\frac{1}{n}\right]^{N}\right)}\right)\widehat{Z}_{\lambda}^{(r)}\left(nx-k\right)-\sum_{k=-\infty}^{\infty}f\left(x\right)\widehat{Z}_{\lambda}^{(r)}\left(nx-k\right)\right|$$

$$=\left|\sum_{k=-\infty}^{\infty}\left(\left(\frac{(C)\int_{\left[0,\frac{1}{n}\right]^{N}}f\left(t+\frac{k}{n}\right)d\mu\left(t\right)}{\mu\left(\left[0,\frac{1}{n}\right]^{N}\right)}\right)-f\left(x\right)\right)\widehat{Z}_{\lambda}^{(r)}\left(nx-k\right)\right|\leq \tag{48}$$

$$\sum_{k=-\infty}^{\infty}\left|\left(\frac{(C)\int_{\left[0,\frac{1}{n}\right]^{N}}f\left(t+\frac{k}{n}\right)d\mu\left(t\right)}{\mu\left(\left[0,\frac{1}{n}\right]^{N}\right)}\right)-f\left(x\right)\right|\widehat{Z}_{\lambda}^{(r)}\left(nx-k\right)\overset{(45)}{\leq}$$

$$\sum_{k=-\infty}^{\infty}\left(\frac{(C)\int_{\left[0,\frac{1}{n}\right]^{N}}\left|f\left(t+\frac{k}{n}\right)-f\left(x\right)\right|d\mu\left(t\right)}{\mu\left(\left[0,\frac{1}{n}\right]^{N}\right)}\right)\widehat{Z}_{\lambda}^{(r)}\left(nx-k\right)=$$

$$\sum_{\substack{k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\infty}\left(\frac{(C)\int_{\left[0,\frac{1}{n}\right]^{N}}\left|f\left(t+\frac{k}{n}\right)-f\left(x\right)\right|d\mu\left(t\right)}{\mu\left(\left[0,\frac{1}{n}\right]^{N}\right)}\right)\widehat{Z}_{\lambda}^{(r)}\left(nx-k\right)+ \tag{49}$$

$$\sum_{\substack{k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty}\left(\frac{(C)\int_{\left[0,\frac{1}{n}\right]^{N}}\left|f\left(t+\frac{k}{n}\right)-f\left(x\right)\right|d\mu\left(t\right)}{\mu\left(\left[0,\frac{1}{n}\right]^{N}\right)}\right)\widehat{Z}_{\lambda}^{(r)}\left(nx-k\right)\leq$$

$$\sum_{\substack{k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\infty}\left(\frac{(C)\int_{\left[0,\frac{1}{n}\right]^{N}}\omega_{1}\left(f,\left\|t\right\|_{\infty}+\left\|\frac{k}{n}-x\right\|_{\infty}\right)d\mu\left(t\right)}{\mu\left(\left[0,\frac{1}{n}\right]^{N}\right)}\right)\widehat{Z}_{\lambda}^{(r)}\left(nx-k\right)+$$

$$2\left\|f\right\|_{\infty}\left(\sum_{\left\{\begin{array}{c} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{array}\right.}^{\infty} \widehat{Z}_{\lambda}^{(r)}\left(|nx-k|\right)\right)$$

$$\overset{(35)}{\leq} \omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+2\left\|f\right\|_{\infty}\ {}_{\lambda}\widehat{\delta}_{N}^{(r)}\left(\beta,n\right), \tag{50}$$

proving the claim. ∎

Additionally we give

**Definition 4.2.** *Denote* $C_B^{+}\left(\mathbb{R}^N,\mathbb{C}\right)=\{f:\mathbb{R}^N\to\mathbb{C}|f=f_1+if_2,$ *where* $f_1,f_2\in C_B^{+}\left(\mathbb{R}^N\right)\}$. *We set for* $f\in C_B^{+}\left(\mathbb{R}^N,\mathbb{C}\right)$ *that*

$${}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f,x\right):={}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f_1,x\right)+i\ {}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f_2,x\right), \tag{51}$$

$\forall\ n\in\mathbb{N},\ x\in\mathbb{R}^N,\ i=\sqrt{-1};\ r,\lambda\in\mathbb{N}.$

We give

**Theorem 4.2.** *Let* $f\in C_B^{+}\left(\mathbb{R}^N,\mathbb{C}\right)$, $f=f_1+if_2$, $N\in\mathbb{N}$, $0<\beta<1$, $x\in\mathbb{R}^N$, $r,\lambda,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

*i)*

$$\sup_{\mu}\left|{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f,x\right)-f\left(x\right)\right|\leq\left(\omega_1\left(f_1,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\omega_1\left(f_2,\frac{1}{n}+\frac{1}{n^{\beta}}\right)\right)$$

$$+2\left(\left\|f_1\right\|_{\infty}+\left\|f_2\right\|_{\infty}\right)_{\lambda}\widehat{\delta}_{N}^{(r)}\left(\beta,n\right)=:\ {}_{\lambda}\varphi_{n}^{(r)}, \tag{52}$$

*and*

*ii)*

$$\sup_{\mu}\left\|{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f\right)-f\right\|_{\infty}\leq\ {}_{\lambda}\varphi_{n}^{(r)}.$$

*Proof.* We have that

$$\left|{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f,x\right)-f\left(x\right)\right|=\left|{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f_1,x\right)+i\ {}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f_2,x\right)-f_1\left(x\right)-if_2\left(x\right)\right|=$$

$$\left|\left({}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f_1,x\right)-f_1\left(x\right)\right)+i\left({}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f_2,x\right)-f_2\left(x\right)\right)\right|\leq$$

$$\left|{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f_1,x\right)-f_1\left(x\right)\right|+\left|{}_{\lambda}^{(r)}\widehat{K}_{n}^{\mu}\left(f_2,x\right)-f_2\left(x\right)\right|\overset{(46)}{\leq}$$

$$\left(\omega_1\left(f_1,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+2\left\|f_1\right\|_{\infty}\ {}_{\lambda}^{(r)}\widehat{\delta}_{N}\left(\beta,n\right)\right)+ \tag{53}$$

$$\left(\omega_1\left(f_2, \frac{1}{n}+\frac{1}{n^\beta}\right)+2\left\|f_2\right\|_\infty \ {}_\lambda^{(r)}\widehat{\delta}_N(\beta, n)\right),$$

proving the claim. ∎

We need

**Definition 4.3.** *Let $\mathcal{L}^*$ be the Lebesgue $\sigma$-algebra on $\mathbb{R}$, and the set function $\mu^*: \mathcal{L}^* \to [0,+\infty]$, which is assumed to be monotone, submodular and strictly positive. For $f \in C_B^+(\mathbb{R})$, we define the infinitely many specific multi-composite univariate Kantorovich-Choquet type neural network operator for any $x \in \mathbb{R}$:*

$$ {}_\lambda^{(r)}\widehat{M}_n^{\mu^*}(f, x)=\sum_{k=-\infty}^{\infty}\left(\frac{(C)\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right) d\mu^*(t)}{\mu^*\left(\left[0, \frac{1}{n}\right]\right)}\right) T_\lambda^{(r)}(nx-k). \tag{54}$$

*Clearly here $\mu^*\left(\left[0, \frac{1}{n}\right]\right)>0$, $\forall\ n \in \mathbb{N}$; $r, \lambda \in \mathbb{N}$.*

*Above we notice that*

$$\left\|{}_\lambda^{(r)}\widehat{M}_n^{\mu^*}(f)\right\|_\infty \leq\|f\|_\infty, \tag{55}$$

*so that ${}_\lambda^{(r)}\widehat{M}_n^{\mu^*}(f, x)$ is well-defined.*

*Notice that ${}_\lambda^{(r)}\widehat{K}_n^{\mu}$, when $N=1$, collapses to ${}_\lambda^{(r)}\widehat{M}_n^{\mu^*}$.*

It follows another related result.

**Corollary 4.1.** *(to Theorem 4.1 when $N=1$)*

*Let $f \in C_B^+(\mathbb{R})$, $0<\beta<1$, $x \in \mathbb{R}$; $r, \lambda, n \in \mathbb{N}$ with $n^{1-\beta}>2$. Then*

*i)*

$$\begin{aligned}\sup_{\mu^*}\left|{}_\lambda^{(r)}\widehat{M}_n^{\mu^*}(f, x)-f(x)\right| &\leq \omega_1\left(f, \frac{1}{n}+\frac{1}{n^\beta}\right)+2\|f\|_\infty\left(1-\widetilde{G}_\lambda^{(r)}\left(n^{1-\beta}-2\right)\right) \\ &=: {}_\lambda\tau_n^{(r)},\end{aligned} \tag{56}$$

*and*

*ii)*

$$\sup_\mu\left\|{}_\lambda^{(r)}\widehat{M}_n^{\mu^*}(f)-f\right\|_\infty \leq {}_\lambda\tau_n^{(r)}. \tag{57}$$

*Given that $f \in \left(C_U^+(\mathbb{R}) \cap C_B^+(\mathbb{R})\right)$, we obtain $\lim_{n \to \infty} {}_\lambda^{(r)}\widehat{M}_n^{\mu^*}(f)=f$, uniformly.*

*Proof.* As similar to Theorem 4.1 is omitted. ∎

We need

**Definition 4.4.** *Let $f \in C_B^+(\mathbb{R},\mathbb{C})$ where $f = f_1 + if_2$ with $f_1, f_2 \in C_B^+(\mathbb{R})$. We set*

$$ {}_{\lambda}^{(r)}\widehat{M}_n^{\mu^*}(f,x) := {}_{\lambda}^{(r)}\widehat{M}_n^{\mu^*}(f_1,x) + i\, {}_{\lambda}^{(r)}\widehat{M}_n^{\mu^*}(f_2,x), \tag{58}$$

$\forall\ n \in \mathbb{N},\ x \in \mathbb{R};\ r, \lambda \in \mathbb{N}$.

We continue with

**Corollary 4.2.** *(to Theorem 4.2 when $N = 1$) Let $f \in C_B^+(\mathbb{R},\mathbb{C})$, $f = f_1 + if_2$, $0 < \beta < 1$, $x \in \mathbb{R}$, $r, \lambda, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

*i)*

$$\sup_{\mu^*}\left| {}_{\lambda}^{(r)}\widehat{M}_n^{\mu^*}(f,x) - f(x)\right| \leq \left(\omega_1\left(f_1, \frac{1}{n} + \frac{1}{n^\beta}\right) + \omega_1\left(f_2, \frac{1}{n} + \frac{1}{n^\beta}\right)\right)$$

$$+2\left(\|f_1\|_\infty + \|f_2\|_\infty\right)\left(1 - \widetilde{G}_\lambda^{(r)}\left(n^{1-\beta} - 2\right)\right) =: {}_{\lambda}\psi_n^{(r)}, \tag{59}$$

*and*

*ii)*

$$\sup_{\mu^*}\left\| {}_{\lambda}^{(r)}\widehat{M}_n^{\mu^*}(f) - f\right\|_\infty \leq {}_{\lambda}\psi_n^{(r)}.$$

*Proof.* As similar to Theorem 4.2 is omitted. ∎

We finish with $L_p$ estimates.

**Theorem 4.3.** *Let all as in Theorem 4.2, $p > 1$, $\widehat{\Lambda}$ a compact subset of $\mathbb{R}^N$. Then*

$$\left\| {}_{\lambda}^{(r)}\widehat{K}_n^{\mu}(f) - f\right\|_{p,\widehat{\Lambda}} \leq {}_{\lambda}\varphi_n^{(r)}\left|\widehat{\Lambda}\right|^{\frac{1}{p}}, \tag{60}$$

*where $\left|\widehat{\Lambda}\right| < \infty$, is the Lebesgue measure of $\widehat{\Lambda}$, where ${}_{\lambda}\varphi_n^{(r)}$ as in (52).*

*Proof.* By (52), etc. ∎

**Theorem 4.4.** *All as in Theorem 4.3, $N = 1$ case. Then*

$$\left\| {}_{\lambda}^{(r)}\widehat{M}_n^{\mu^*}(f) - f\right\|_{p,\widehat{\Lambda}} \leq {}_{\lambda}\psi_n^{(r)}\left|\widehat{\Lambda}\right|^{\frac{1}{p}}, \tag{61}$$

*where $\widehat{\Lambda}$ now is a closed interval of $\mathbb{R}$, where ${}_{\lambda}\psi_n^{(r)}$ as in (59).*

*Proof.* By (59), etc. ∎

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# SOME NEW DIFFERENCE CESÀRO SEQUENCE SPACES

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**Abstract** In this paper, we define some new Cesaro sequence spaces using $\Delta_v^m$. Furthermore we give some topological properties of these sequence spaces.

**Keywords:** Cesaro sequence space, difference operator, generalized difference sequence spaces.
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## 1. INTRODUCTION

The origin of Cesaro sequence spaces lies in the methods of convergence through averaging in analysis. The aim is to make convergent some disconvergent sequences by averaging them. Cesaro array spaces were defined by Shiue [1] as follows:

$$ces_p = \left\{ x = (x_k) \in \omega : \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\} (1 < p < \infty)$$

and

$$ces_\infty = \left\{ x = (x_k) \in \omega : \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}.$$

Lim [2] defined this space with a different norm as follows:

$$Ces_p = \left\{ x = (x_k) \in \omega : \left( \sum_{r=0}^{\infty} \left( \frac{1}{2^r} \sum_{r} |x_k| \right)^p \right)^{1/p} < \infty \right\} (1 < p < \infty)$$

and

$$Ces_\infty = \left\{ x = (x_k) \in \omega : \sup_{r \geq 0} \frac{1}{2^r} |x_r| < \infty \right\}$$

where $\sum_r$ represents a sum over the interval $[2^r, 2^r + 1)$.

Ng and Lee [3] defined Cesaro sequence spaces of non-absolute type:

$$X_p = \left\{ x = (x_k) \in \omega : \left( \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty \right\} (1 < p < \infty)$$

and

$$X_\infty = \left\{ x = (x_k) \in \omega : \left| \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty \right\}.$$

These spaces have the norms, respectively,

$$\|x\|_p = \left( \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p}$$

$$\|x\|_\infty = \sup \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| ; \; n \in N \right\}$$

and are Banach spaces.

Consequently, the study of Cesaro sequence spaces brings information on how sequences behave under averaging. It is fundamental to summability theory because some non-convergent sequences become convergent by averaging them. Furthermore, the use of Cesaro averaging in conjunction with difference operators has led to the emergence of new sequence spaces of the generalized difference Cesaro type in the literature, demonstrating how extensible and flexible this structure is.

In this context, difference operators and the difference sequence spaces associated with them will be examined. The relationships between the Cesaro structure and the difference structure will be made clearer. Difference sequence spaces examine the convergence or boundedness properties of a sequence obtained by taking the differences between the terms of a sequence. The first-order difference operator $\Delta$, a fundamental tool in sequence theory, produces the sequence $\Delta x$ when applied to the sequence $x = (x_k)$. The first-order difference space $\Delta(X)$ is defined by Kızmaz [4] as follows:

$$\Delta x_k = x_k - x_{k+1} \quad (k \in N)$$

and $X = l_\infty, c$ and $c_0$

$$\Delta(X) = \{x = (x_k) \in \omega : (\Delta x_k) \in X\}$$

After,· Et and Çolak [5] defined the sequence spaces

$$\Delta^m(X) = \{x = (x_k) \in \omega : (\Delta^m x_k) \in X\}$$

for $X = l_\infty, c$ and $c_0$ such that $m \in N$, $\Delta^0 x_k = x_k$

$$\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$$

$$\Delta^m x_k = \sum_{j=0}^{m} (-1)^j \binom{m}{j} x_{k+j}.$$

The classical difference operator only takes successive differences; however, for more detailed behaviors of sequences, a generalized difference operator is needed.

Et and Esi [6] defined the sequence spaces

$$\Delta_v^m(X) = \{x = (x_k) : (\Delta_v^m x_k) \in X\},$$

where $v = (v_k)$ is a sequence of non-zero real numbers, $m \in N$, $\Delta_v^0 x_k = v_k x_k$, $\Delta_v^m x_k = \Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1}$ and

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i} \quad (k \in N),$$

for $X = l_\infty, c, c_0$. The generalized difference Cesaro sequence spaces $EC_p(\Delta^m)$ and $EC_\infty(\Delta^m)$ discussed in this study are improved and extended versions of the sequence space defined in the article prepared by Braha et al.[7]. These spaces have allowed for a more comprehensive definition of the classical $l_p$, $c$, $c_0$ sequence spaces by re-examining them with the help of Cesaro mean and difference operators. Therefore, the generalized difference Cesaro sequence spaces $EC_p(\Delta^m)$ and $EC_\infty(\Delta^m)$, defined below, have paved the way for the emergence of new topological problems.

**Definition 1.1.** **[7]** *For $m \geq 0$, $0 < p < \infty$, the sequence space $EC_p(\Delta^m)$ is defined as;*

$$EC_p(\Delta^m) = \left\{ x = (x_s) : \sum_{s=1}^{\infty} \left| \frac{1}{2^s} \sum_{j=0}^{s} \binom{s}{j} \frac{1}{j+1} \sum_{u=0}^{j} \Delta^m x_u \right|^p < \infty \right\}.$$

In this definition, the generalized difference operator is first applied to the sequence, then the resulting sequence is averaged with the Cesaro operator, and finally evaluated with the $l_p$ norm. On the $EC_p(\Delta^m)$ space, the expression

$$\|x\|_{EC_p} = \sum_{i=1}^{m} |x_i| + \left( \sum_{s=1}^{\infty} \left| \frac{1}{2^s} \sum_{j=0}^{s} \binom{s}{j} \frac{1}{j+1} \sum_{u=0}^{j} \Delta^m x_u \right|^p \right)^{1/p}$$

defines a norm. In the literature, it has been shown that under this norm the space is complete, and therefore a Banach sequence space [7].

**Definition 1.2.** **[7]** *Let $m \geq 0$ and $0 < p < \infty$ . Then $EC_\infty(\Delta^m)$ is defined as follows*:

$$EC_\infty(\Delta^m) = \left\{ x = (x_s) : \sup_s \left| \frac{1}{2^s} \sum_{j=0}^{s} \binom{s}{j} \frac{1}{j+1} \sum_{u=0}^{j} \Delta^m x_u \right| < \infty \right\}.$$

In this definition, after taking the generalized difference of the sequence, the Cesaro mean is calculated, and the condition that the resulting sequence must be bounded is sought. Therefore, the space $EC_\infty(\Delta^m)$ can be interpreted as the generalized difference Cesaro equivalent of the classical $l_\infty$ space.

The expression

$$\|x\|_{EC_\infty} = \sum_{i=1}^{m} |x_i| + \sup_s \left| \frac{1}{2^s} \sum_{j=0}^{s} \binom{s}{j} \frac{1}{j+1} \sum_{u=0}^{j} \Delta^m x_u \right|$$

defines a norm on the space $EC_\infty(\Delta^m)$. In the literature, it has been shown that under this norm, the space $EC_\infty(\Delta^m)$ is complete, and therefore constitutes a Banach sequence space. Furthermore, since the continuity of the coordinate functionals is ensured, this space has the property of being a BK-space [7].

## 2. Main Results

Using the information given here, new sequence spaces for the generalized difference operator, $EC_p(\Delta_v^m, u)$ and $EC_\infty(\Delta_v^m, u)$, will be defined and some topological properties will be given.

**Definition 2.1.** *Let $m \geq 0$, $0 < p < \infty$, $u = (u_k)$ and $v = (v_k)$ be a sequence of non-zero real numbers. The sequence spaces $EC_p(\Delta_v^m, u)$ and $EC_\infty(\Delta_v^m, u)$ are defined as follows*:

$$EC_p(\Delta_v^m, u) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \left| \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} u_s \Delta_v^m x_s \right|^p < \infty \right\},$$

$$EC_\infty(\Delta_v^m, u) = \left\{ x = (x_k) : \sup_k \left| \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} u_s \Delta_v^m x_s \right| < \infty \right\}.$$

**Theorem 2.2.** *$EC_p(\Delta_v^m, u)$ sequence space is a Banach space with norm*

$$\|x\|_{EC_p} = \sum_{i=1}^{m} |x_i| + \left( \sum_{k=1}^{\infty} \left| \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} u_s \Delta_v^m x_s \right|^p \right)^{1/p}. \quad (1)$$

*The space $EC_\infty(\Delta_v^m, u)$ is a Banach space with norm*

$$\|x\|_{EC_\infty} = \sum_{i=1}^{m} |x_i| + \sup_k \left| \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} u_s \Delta_v^m x_s \right|. \quad (2)$$

*Proof.* We will prove the theorem for the space $EC_\infty(\Delta_v^m, u)$ since the proof is similar for $EC_p(\Delta_v^m, u)$. Let $(x^r) = (x_1^r, x_2^r, ..., x_n^r, ...) \in EC_\infty(\Delta_v^m, u)$ be a Cauchy sequence for each $r \in N$. Then

$$\|x^r - x^q\|_{EC_\infty} = \sum_{i=1}^{m} |x_i^r - x_i^q| + \sup_k \left| \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} u_s \Delta_v^m (x_s^r - x_s^q) \right| \to 0.$$

From this, for any $i \in N$, if $r \to \infty$ and $q \to \infty$, we obtain $|x_s^r - x_s^q| \to 0$. Therefore, the sequence $(x^r) = (x_1^r, x_2^r, ..., x_n^r, ...)$ is a Cauchy sequence in $C$ and is therefore convergent. Let $\lim_r x_i^r = x_i$ for $\forall i \in N$. Since $(x^r)$ is a Cauchy sequence for each $\varepsilon > 0$, for $\forall r, q \geq n_0$,

$$\sum_{i=1}^{m} |x_i^r - x_i^q| \leq \varepsilon$$

and

$$\left| \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} u_s \Delta_v^m (x_s^r - x_s^q) \right| \leq \varepsilon$$

holds. From the last relations, for $\forall n \geq n_0$

$$\lim_{q\to\infty} \sum_{i=1}^{m} |x_i^r - x_i^q| = \sum_{i=1}^{m} |x_i^r - x_i| \leq \varepsilon$$

$$\lim_{q\to\infty} \left| \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} u_s \Delta_v^m (x_s^r - x_s^q) \right| =$$

$$= \left| \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} u_s \Delta_v^m (x_s^r - x_s) \right| \leq \varepsilon$$

is obtained. This means that $\|x^r - x\|_{EC_\infty} \leq \varepsilon$ for $\forall r \geq n_0$, and it means that $x^r \to x$ as $r \to \infty$. In addition to this,

$$\begin{aligned}
\left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j+1}\sum_{s=0}^{j}u_s\Delta_v^m(x_s^r)\right| &= \left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{\sum_{s=0}^{j}u_s\Delta_v^m(x_s^r + x_s^{n_0} - x_s^{n_0})}{j+1}\right| \\
&\leq \left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j+1}\sum_{s=0}^{j}u_s\Delta_v^m(x_s^r - x_s^{n_0})\right| \\
&+ \left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j+1}\sum_{s=0}^{j}u_s\Delta_v^m(x_s^{n_0})\right| \\
&\leq \|x - x^{n_0}\|_{EC_\infty} + \\
&+ \left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j+1}\sum_{s=0}^{j}u_s\Delta_v^m(x_s^{n_0})\right| \\
&< \infty
\end{aligned}$$

is obtained. From this, $x \in EC_\infty(\Delta_v^m, u)$ and $EC_\infty(\Delta_v^m, u)$ is a Banach space. Similarly, it can be shown that $EC_p(\Delta_v^m, u)$ is a Banach space with the norm given by (1).

**Result 2.3.** *The spaces $EC_p(\Delta_v^m, u)$ and $EC_\infty(\Delta_v^m, u)$ are $BK-$ spaces under the norms defined in (1) and (2), respectively.*

**Theorem 2.4.** *If $1 \leq p < q$ then $EC_p(\Delta_v^m, u) \subset EC_q(\Delta_v^m, u)$.*

*Proof.* Since

$$\left(\sum_{k=1}^{n}|a_k|^q\right)^{1/q} \leq \left(\sum_{k=1}^{n}|a_k|^p\right)^{1/p}$$

the proof is obtained easily.

**Theorem 2.5.**

1 *The inclusion $EC_\infty(\Delta_v^{m-1}, u) \subset EC_\infty(\Delta_v^m, u)$ is strict.*

2 *Let $p \in R^+$ such that $1 \leq p < q$.*
*Then, the inclusion $EC_p(\Delta_v^{m-1}, u) \subset EC_p(\Delta_v^m, u)$ is strict.*

*Proof.* i) It can be easily shown that $EC_\infty(\Delta_v^{m-1}, u) \subset EC_\infty(\Delta_v^m, u)$ is true. To show that it is strict, let's take m=2, $(u_k) = (1)$, $(v_k) = (1)$

and $x = (x_k) = (k^2)$. Then $(k^2) \in EC_\infty(\Delta_v^m, u) \backslash EC_\infty(\Delta_v^{m-1}, u)$ is true. Indeed;

$$\begin{aligned}
\left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j+1}\sum_{s=0}^{j}\Delta_v x_s\right| &= \left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j+1}(-x_{j+1})\right| \\
&= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(j+1) \\
&= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}j + \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} \\
&= \left(\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}j\right)+1 \\
&= \left(\frac{1}{2^k}\sum_{j=0}^{k}k\binom{k-1}{j-1}\right)+1 \\
&= \frac{k2^{k-1}}{2^k}+1 = \frac{k+2}{2} \to \infty \ \ (k \to \infty)\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j+1}\sum_{s=0}^{j}\Delta_v^2 x_s\right| &= \left|\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j+1}(-\Delta x_{j+1})\right| \\
&= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j+1}(2j+3) \\
&= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(2+\frac{1}{j+1}\right) \\
&= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(2+\frac{1}{j+1}\right) \\
&= \frac{2}{2^k}\sum_{j=0}^{k}\binom{k}{j} + \frac{1}{2^k}\sum_{j=0}^{k}\frac{\binom{k}{j}}{j+1} \\
&< \frac{2}{2^k}\sum_{j=0}^{k}\binom{k}{j} + \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} = 2+1 = 3.
\end{aligned}$$

ii) The other can be proven similarly.

**Theorem 2.6.** *$EC_p$ and $EC_\infty$ are solid and therefore monotone. However, $EC_p(\Delta_v^m, u)$ and $EC_\infty(\Delta_v^m, u)$ are neither solid, nor symmetric, nor sequence algebras (for $m \geq 1$).*

*Proof.* Let $x = (x_k) \in EC_\infty$ and $y = (y_k)$ be a sequence such that for every $k \in N$, $|y_k| \leq |x_k|$. Then we obtain

$$\left| \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} y_s \right| < \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \sum_{s=0}^{j} |x_s|.$$

From this, we obtain that $EC_\infty$ is solid and monotone. Let m=2.Then $(x_k) = (k^2) \in EC_\infty(\Delta^2)$, but $\alpha_k x_k \notin EC_\infty(\Delta^2)$ when $\alpha_k = (-1)^k$ for $\forall k \in N$. Therefore, $EC_\infty(\Delta^2)$ is not solid.

The other cases can be seen with the help of similar examples.

### 3. Conclusions

In this study, the sequence spaces $EC_p(\Delta_v^m, u)$ and $EC_\infty(\Delta_v^m, u)$, defined by the joint application of generalized difference and Cesaro operators, are investigated from topological perspectives. These spaces can be interpreted as generalizations that unify Cesaro sequence spaces, generalized difference sequence spaces, and classical $\ell_p$ spaces which are often treated separately in the literature under a single framework.

It is shown that suitable norms can be defined for both spaces, and their completeness under these norms is established, thereby proving that they are Banach sequence spaces.

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# SOME NEW $\lambda$-WEAK CONVERGENT SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

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**Abstract** In this article, we will give some new $\lambda$-weak convergent sequence spaces defined by a sequence of Orlicz functions. We will examine the algebraic and topological properties of these newly defined sequence spaces and give some inclusion relations between them.

**Keywords:** weak convergence, Orlicz function, $\lambda-$convergence, difference sequence.
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## 1. INTRODUCTION

Mursaleen [1] extended the concept of natural density to the concept of $\lambda-$density and defined $\lambda-$statistical convergence. If $\lambda = (\lambda_s)$ is an infinite, non-decreasing sequence of positive real numbers satisfying $\lambda_1 = 1$ and $\lambda_{s+1} < \lambda_s + 1$, then for any subset $T \subset \mathbb{N}$, the $\lambda-$density $d_\lambda(T)$ is defined as

$$d_\lambda(T) = \lim_{s\to\infty} \frac{|\{t \in I_s : t \in T\}|}{\lambda_s}$$

where $I_s = [s - \lambda_s + 1, s]$.

A sequence $t = (t_\alpha)$ of real numbers is called $\lambda-$statistically convergent to $t_0 \in \mathbb{R}$, if for every $\varepsilon > 0$, $d_\lambda(T(\varepsilon)) = 0$ where $T(\varepsilon) = \{\alpha \in \mathbb{N} : |t_\alpha - t_0| \geq \varepsilon\}$.

Firstly Kizmaz [2] defined the difference sequence concept and Et and Çolak [3] generalized this idea. Further, Et and Esi [4] defined the following sequence spaces using by any fixed sequence of nonzero complex numbers $v = (v_\alpha)$:

$$Z(\Delta_v^p) = \{t = (t_\alpha) \in \omega : \Delta_v^p t \in Z\}$$

for $Z = l_\infty$, $c$ and $c_0$; where $\Delta_v^p t_\alpha = \Delta_v^{p-1} t_\alpha - \Delta_v^{p-1} t_{\alpha+d}$ and $\Delta_v^0 t_\alpha = t_\alpha$ such that

$$\Delta_v^p t_\alpha = \sum_{d=0}^{p} (-1)^d \binom{p}{d} v_{\alpha+d} t_{\alpha+d}$$

for all $\alpha \in \mathbb{N}$. The set of all convex combinations of $u$ and $v$ is the set of points

$$Y = \{w_\varrho \in V : w_\varrho = (1-\varrho)\, u + \varrho v, 0 \leq \varrho \leq 1\}$$

where $V$ is a real vector space and $u, v \in V$. The set $M \subset V$ is said to be convex if $Y \subset M$ for any two points $u, v \in M$.

An Orlicz function $U : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing, and convex function satisfying the conditions $U(0) = 0$, $U(t) > 0$ for $t > 0$ and $U(t) \to \infty$ as $t \to \infty$.

Lindenstrauss and Tzafriri [5] defined Orlicz function sequence space

$$l_U = \left\{ (t_i) \in w : \sum_{i=1}^{\infty} U\left(\frac{|t_i|}{\mu}\right) < \infty \text{ for some } \mu > 0 \right\},$$

with the norm

$$\|t\| = \inf \left\{ \mu > 0 : \sum_{i=1}^{\infty} U\left(\frac{|t_i|}{\mu}\right) \leq 1 \right\}.$$

Let $X$ be a normed linear space. A sequence $(t_s) \subset X$ is called weakly convergent to $t_0 \in X$ if

$$\lim_{s\to\infty} f(t_s - t_0) = 0, \ \forall f \in X'.$$

Here $X'$ is the continuous dual space of $X$.

A sequence $(t_i) \subset X$ is called $\lambda-$weakly convergent to $t_0 \in X$ if

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{i \in I_s} f(t_i - t_0) = 0, \ \forall f \in X'.$$

The set of all $\lambda-$weakly convergent sequences is denoted by $D_\lambda^\omega$ [6].

A sequence space $E$ is called solid if $(\beta_i t_i) \in E$, for each $(t_i)$ in $E$ and each scalar sequence $(\beta_i)$ with $|\beta_i| \leq 1$, for all $i \in \mathbb{N}$.

A sequence space $E$ is called monotone if it contains all preimages of its step spaces and symmetric if $\left(t_{\pi(i)}\right) \in E$ whenever $(t_i) \in E$ and $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation of $\mathbb{N}$.

An Orlicz function $U$ satisfies the $\Delta_2-$condition if there is a constant $T > 0$ such that for all $u \geq 0$

$$U\left(2u\right) \leq TU\left(u\right).$$

Kişi and Gürdal [6] defined some sequence spaces by using an Orlicz function and showed that these spaces are linear spaces.

## 2. MAIN RESULTS

In this section, we define some sequence spaces by using Orlicz functions sequences and showed linear spaces them.

**Definition 2.1.** *Let $U = (U_\alpha)$ be a sequence of Orlicz functions, $v = (v_t)$ be any fixed sequence of nonzero complex numbers and $p \in \mathbb{N}$. Then we define*

$$[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0 = \left\{ t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{\left|f\left(\Delta_v^p t_\alpha\right)\right|}{\mu} \right) = 0 \text{ for some } \mu > 0 \right\}$$

$$[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_1 = \left\{ t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{\left|f\left(\Delta_v^p t_\alpha - t_0\right)\right|}{\mu} \right) = 0 \text{ for some } t_0 \text{ and } \mu > 0 \right\}$$

$$[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_\infty = \left\{ t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{\left|f\left(\Delta_v^p t_\alpha\right)\right|}{\mu} \right) < \infty \text{ for some } \mu > 0 \right\}.$$

**Theorem 2.2.** *The sequence spaces $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0$, $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_1$ and $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_\infty$ are linear spaces.*

*Proof.* Since it's similar for other cases as well, we'll only prove it for $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0$. Let $(t_\alpha), (q_\alpha) \in [D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0$ and $\eta, \xi \in C$. We need to find some $\mu > 0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{\left|f\left(\eta\Delta_v^p t_\alpha + \xi\Delta_v^p q_\alpha\right)\right|}{\mu} \right) = 0.$$

Since $(t_\alpha), (q_\alpha) \in [D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0$ there exist $\mu_1, \mu_2 > 0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{\left|f\left(\Delta_v^p t_\alpha\right)\right|}{\mu_1} \right) = 0,$$

$$\lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}U_\alpha\left(\frac{|f(\Delta_v^p q_\alpha)|}{\mu_2}\right)=0.$$

If we say $A_\alpha=\frac{|f(\eta\Delta_v^p t_\alpha)|}{\mu_1}$, $B_\alpha=\frac{|f(\xi\Delta_v^p q_\alpha)|}{\mu_2}$, $\mu=\mu_1+\mu_2$, $v_1=\frac{\mu_1}{\mu}$, $v_2=\frac{\mu_2}{\mu}$,

then we have

$$U_\alpha\left(\left(\mu_1A_\alpha+\mu_2B_\alpha\right)/\mu\right)=U_\alpha\left(v_1A_\alpha+v_2B_\alpha\right)\le v_1U_\alpha\left(A_\alpha\right)+v_2U_\alpha\left(B_\alpha\right)$$

and hence if we take sum from 1 to infinity, then divide by $\lambda_s$ and take the limit as $s\to\infty$, we obtain the result. ∎

**Corollary 2.3.** *The sequence spaces* $[D_\lambda^\omega,\mathcal{U},\Delta_v^p]_0$, $[D_\lambda^\omega,\mathcal{U},\Delta_v^p]_1$ *and* $[D_\lambda^\omega,\mathcal{U},\Delta_v^p]_\infty$ *are convex.*

**Theorem 2.4.** *Let $U=(U_\alpha)$ be any sequence of Orlicz functions. Then the space $[D_\lambda^\omega,\mathcal{U},\Delta_v^p]_\infty$ is a normed linear space with norm*

$$X\Delta_v^p(t)=\sum_{i=1}^{p}|f(x_i)|+\inf\left\{\mu>0:\sup_s\frac{1}{\lambda_s}\sum_{\alpha\in I_s}U_\alpha\left(\frac{|f(\Delta_v^p t_\alpha)|}{\mu}\right)\le 1\right\}.$$

*Proof.* Clearly $X\Delta_v^p(t)=X\Delta_v^p(-t)$ and $X\Delta_v^p(t)=0\Leftrightarrow t=\theta$. Now let's show the subadditivity of $X\Delta_v^p(t)$. Let $\mu_1,\mu_2>0$ be such that

$$\sup_s\frac{1}{\lambda_s}\sum_{\alpha\in I_s}U_\alpha\left(\frac{|f(\Delta_v^p t_\alpha)|}{\mu_1}\right)\le 1$$

$$\sup_s\frac{1}{\lambda_s}\sum_{\alpha\in I_s}U_\alpha\left(\frac{|f(\Delta_v^p \varpi_\alpha)|}{\mu_2}\right)\le 1$$

and say $\mu=\mu_1+\mu_2$. Then we have

$$\sup_s\frac{1}{\lambda_s}\sum_{\alpha\in I_s}U_\alpha\left(\frac{|f(\Delta_v^p(t_\alpha+\varpi_\alpha))|}{\mu}\right)\le 1.$$

Since $\mu$ is nonnegative, we have

$$
\begin{aligned}
X\Delta_v^p(t+\varpi) &= \sum_{i=1}^{p} |f(t_i+\varpi_i)| \\
&+ \inf\left\{\mu>0 : \sup_s \tfrac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha\left(\tfrac{\left|f\left(\Delta_v^p(t_\alpha+\varpi_\alpha)\right)\right|}{\mu}\right) \le 1\right\},
\end{aligned}
$$

and hence

$$X\Delta_v^p(t+\varpi) \le X\Delta_v^p(t) + X\Delta_v^p(\varpi).$$

Let $\gamma \neq 0$ be a complex number. Then we have

$$
\begin{aligned}
X\Delta_v^p(\gamma t) &= \sum_{i=1}^{p} |f(\gamma t_i)| + \inf\left\{\mu>0 : \sup_s \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha\left(\frac{|f(\Delta_v^p(\gamma t_\alpha))|}{\mu}\right) \le 1\right\} \\
&\le |\gamma|\, X\Delta_v^p(t).
\end{aligned}
$$

∎

**Theorem 2.5.** *Let $U=(U_\alpha)$ and $V=(V_\alpha)$ be sequences of Orlicz functions satisfying the $\Delta_2-$condition. Then the following inclusions are strict for $\mathcal{K}=0,1$ and $\infty$.*

*i)* $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_{\mathcal{K}} \subseteq [D_\lambda^\omega, \mathcal{U}.V, \Delta_v^p]_{\mathcal{K}}$,

*ii)* $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_{\mathcal{K}} \cap [D_\lambda^\omega, V, \Delta_v^p]_{\mathcal{K}} \subseteq [D_\lambda^\omega, \mathcal{U}+V, \Delta_v^p]_{\mathcal{K}}$.

*Proof. i)* We only show for $\mathcal{K}=0$. Let $(t_\alpha) \in [D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_\infty$. Then there exists $\mu>0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha\left(\frac{|f(\Delta_v^p(t_\alpha))|}{\mu}\right) = 0.$$

Let $0<\sigma<1$ and $0<\beta<1$ be such that $V_\alpha(m)<\sigma$ for $0\le m<\beta$. Define

$$\varpi_\alpha = U_\alpha\left(\frac{|f(\Delta_v^p(t_\alpha))|}{\mu}\right).$$

Consider the expression

$$\sum_{\alpha\in I_s} V_\alpha(\varpi_\alpha) = \sum_1 V_\alpha(\varpi_\alpha) + \sum_2 V_\alpha(\varpi_\alpha).$$

Here the first sum is calculated based on $\varpi_\alpha > \beta$ and the second sum is calculated based on $\varpi_\alpha \leq \beta$. Since

$$\frac{1}{\lambda_s}\sum_1 V_\alpha\left(\varpi_\alpha\right) < V_\alpha\left(2\right)\frac{1}{\lambda_s}\sum_1\left(\varpi_\alpha\right) \tag{2.1}$$

for $\varpi_\alpha > \beta$, we have

$$\varpi_\alpha < 1 + \frac{\varpi_\alpha}{\beta}.$$

Since each $V_\alpha$ is nondecreasing and convex and satisfies the $\Delta_2-$condition, it follows that

$$V_\alpha\left(\varpi_\alpha\right) < \frac{1}{2}V_\alpha\left(2\right) + \frac{1}{2}V_\alpha\left(\frac{2\varpi_\alpha}{\beta}\right)$$

$$V_\alpha\left(\varpi_\alpha\right) = T\frac{\varpi_\alpha}{\beta}V_\alpha\left(2\right).$$

Hence,

$$\frac{1}{\lambda_s}\sum_1 V_\alpha\left(\varpi_\alpha\right) \leq \max\left(1, T\beta^{-1}V_\alpha\left(2\right)\right)\frac{1}{\lambda_s}\sum_2\varpi_\alpha. \tag{2.2}$$

If we take the limit as $s \to \infty$, from (2.1) and (2.2), we obtain $(t_\alpha) \in [D_\lambda^\omega, \mathcal{U}.V, \Delta_v^p]_0$.

For other cases, proof can be presented in a similar manner.

*ii)* The proof is omitted. ∎

**Corollary 2.6.** *The inclusion* $[D_\lambda^\omega, \Delta_v^p]_0 \subseteq [D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0$ *is strict. Here, the space* $[D_\lambda^\omega, \Delta_v^p]_0$ *is defined by*

$$[D_\lambda^\omega, \Delta_v^p]_0 = \left\{t = (t_\alpha) : \lim_{s\to\infty}\frac{1}{\lambda_s}\sum_{\alpha\in I_s}\left(\frac{\left|f\left(\Delta_v^p t_\alpha\right)\right|}{\mu}\right) = 0 \text{ for some } \mu > 0\right\}.$$

**Theorem 2.7.** *The inclusion* $\left[D_\lambda^\omega, \mathcal{U}, \Delta_v^{p-1}\right]_\mathcal{K} \subset [D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_\mathcal{K}$ *is strict for* $p \in \mathbb{N}$. *In general,* $\left[D_\lambda^\omega, \mathcal{U}, \Delta_v^i\right]_\mathcal{K} \subset [D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_\mathcal{K}$ *for* $i = 0, 1, 2, ..., p-1$ *and* $\mathcal{K} = 0, 1, \infty$.

*Proof.* If $(t_\alpha) \in \left[D_\lambda^\omega, \mathcal{U}, \Delta_v^{p-1}\right]_0$, then for some $\mu > 0$

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{\left|f\left(\Delta_v^{p-1} t_\alpha\right)\right|}{\mu} \right) = 0. \tag{2.3}$$

Since each $U_\alpha$ is both convex and nondecreasing, we obtain

$$\begin{aligned} \tfrac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \tfrac{\left|f\left(\Delta_v^{p} t_\alpha\right)\right|}{2\mu} \right) &= \tfrac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \tfrac{\left|f\left(\Delta_v^{p-1} t_\alpha - \Delta_v^{p-1} t_{\alpha+1}\right)\right|}{2\mu} \right) \\ &\leq \tfrac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \tfrac{\left|f\left(\Delta_v^{p-1} t_\alpha\right)\right|}{2\mu} \right) \\ &\quad + \tfrac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \tfrac{\left|f\left(\Delta_v^{p-1} t_{\alpha+1}\right)\right|}{2\mu} \right). \end{aligned}$$

As $s \to \infty$, we have

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{\left|f\left(\Delta_v^{p} t_\alpha\right)\right|}{2\mu} \right) = 0$$

by (2.3), which implies $(t_\alpha) \in \left[D_\lambda^\omega, \mathcal{U}, \Delta_v^{p-1}\right]_0$.

By using induction, we can establish that

$$\left[D_\lambda^\omega, \mathcal{U}, \Delta_v^{i}\right]_{\mathcal{K}} \subset \left[D_\lambda^\omega, \mathcal{U}, \Delta_v^{p}\right]_{\mathcal{K}}.$$

If we take $U_\alpha(t) = U(t)$ and $v_\alpha = 1$ for all $\alpha \in \mathbb{N}$, we get Example 1 in [6]. This shows us that the inclusion is strict. ∎

**Theorem 2.8.** *The space $[D_\lambda^\omega, \mathcal{U}, \Delta_v^{p}]_{\mathcal{K}}$, where $K = 0, 1, \infty$, is generally not solid. The spaces $[D_\lambda^\omega, \mathcal{U}]_0$ and $[D_\lambda^\omega, \mathcal{U}]_\infty$ are solid where*

$$[D_\lambda^\omega, \mathcal{U}]_0 = \left\{ t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{|f(t_\alpha)|}{\mu} \right) = 0 \textit{ for some } \mu > 0 \right\},$$

$$[D_\lambda^\omega, \mathcal{U}]_\infty = \left\{ t = (t_\alpha) : \lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{|f(t_\alpha)|}{\mu} \right) < \infty \textit{ for some } \mu > 0 \right\}.$$

*Proof.* If $(t_\alpha) \in [D_\lambda^\omega, \mathcal{U}]_0$, there is $\mu > 0$ such that

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{|f(t_\alpha)|}{\mu} \right) = 0.$$

Let $|\delta_\alpha| \leq 1$. Then

$$\frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{|f(\delta_\alpha t_\alpha)|}{\mu} \right) \leq \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{|f(t_\alpha)|}{\mu} \right),$$

and hence

$$\lim_{s\to\infty} \frac{1}{\lambda_s} \sum_{\alpha\in I_s} U_\alpha \left( \frac{|f(\delta_\alpha t_\alpha)|}{\mu} \right) = 0,$$

which implies that $(\delta_\alpha t_\alpha) \in [D_\lambda^\omega, \mathcal{U}]_0$. The proof for $[D_\lambda^\omega, \mathcal{U}]_\infty$ is similar. ∎

If we take $U_\alpha(t) = U(t)$ and $v_\alpha = 1$ for all $\alpha \in \mathbb{N}$, we can shown that the spaces $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_1$ and $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_\infty$ are generally not solid as Example 2 in [6].

If we take $U_\alpha(t) = U(t)$ and $v_\alpha = 1$ for all $\alpha \in \mathbb{N}$, we can obtain that $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0$ is generally not solid as Example 3 in [6].

**Corollary 2.9.** *The space $[D_\lambda^\omega, \mathcal{U}]_0$ and $[D_\lambda^\omega, \mathcal{U}]_\infty$ are monotone.*

**Remark 2.10.** *The space $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0$ is not convergence free.*

If we take $U_\alpha(t) = U(t)$ and $v_\alpha = 1$ for all $\alpha \in \mathbb{N}$, we can obtain that $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0$ is generally not convergence free as Example 4 in [6].

**Remark 2.11.** *The spaces $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_{\mathcal{K}}$ for $K = 0, 1, \infty$, are generally not symmetric.*

If we take $U_\alpha(t) = U(t)$ and $v_\alpha = 1$ for all $\alpha \in \mathbb{N}$, we can obtain that $[D_\lambda^\omega, \mathcal{U}, \Delta_v^p]_0$ is generally not symmetric as Example 5 in [6].

## 3. CONCLUSIONS

In this paper, we define some $\lambda-$weak convergent sequence spaces by using Orlicz function sequences, and provide an overview of their structures and behaviors by examining some of their algebraic and topological properties in depth. Furthermore, by establishing fundamental coverage relations between these newly defined spaces, particularly in the context of sequence spaces and Orlicz functions, we make a contribution to the field of functional analysis and offer new avenues for researchers working in this area.

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# $G$ METHOD AND FINITE-TIME CONSENSUS

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**Abstract** We give an extension of the $G$ method, with results, the extension and results being partly suggested by the finite Markov chains and specially by the finite-time consensus problem for the DeGroot model and that for the DeGroot model on distributed systems. For the (homogeneous and nonhomogeneous) DeGroot model, using the $G$ method, a result for reaching a partial or total consensus in a finite time is given. Further, we consider a special submodel/case of the DeGroot model, with examples and comments — a subset/subgroup property is discovered. For the DeGroot model on distributed systems, using the $G$ method too, we have a result for reaching a partial or total (distributed) consensus in a finite time similar to that for the DeGroot model for reaching a partial or total consensus in a finite time. Then we show that for any connected graph having $2^m$ vertices, $m \geq 1$, and a spanning subgraph isomorphic to the $m$-cube graph, distributed averaging is performed in $m$ steps — this result can be extended — research work — for any graph with $n_1 n_2 ... n_t$ vertices under certain conditions, where $t, n_1, n_2, ..., n_t \geq 2$, and, in this case, distributed averaging is performed in $t$ steps.

**Keywords:** generalized stable matrix, $G$ method, DeGroot model, consensus problem, finite-time consensus, finite-time partial consensus, subset/subgroup property, distributed system, DeGroot model on distributed systems, finite-time distributed consensus, finite-time partial distributed consensus, finite-time distributed averaging.
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## 1. AN EXTENSION OF $G$ METHOD

In this section, we give an extension of the $G$ method [18]-[19], see also [17] — a good thing both for the forward products of matrices, in particular, for the finite Markov chains, and for the backwards products of matrices, in particular, for the DeGroot model and for the DeGroot model on distributed systems.

Set

$$\mathrm{Par}\,(E) = \{\Delta \mid \Delta \text{ is a partition of } E\},$$

where $E$ is a nonempty set. We shall agree that the partitions do not contain the empty set. $(E)$ is the improper (degenerate) partition of $E$ — we need this partition.

**Definition 1.1.** *Let $\Delta_1, \Delta_2 \in$Par$(E)$. We say that $\Delta_1$ is finer than $\Delta_2$ if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.*

*Write $\Delta_1 \preceq \Delta_2$ when $\Delta_1$ is finer than $\Delta_2$.*

In this article, a vector is a row vector and a stochastic matrix is a row stochastic matrix.

Set

$$\langle m \rangle = \{1, 2, ..., m\} \ \ (m \geq 1).$$

Let $Z$ be an $m \times n$ matrix. $Z_{ij}$ or, if confusion can arise, $Z_{i \to j}$ denotes the entry $(i, j)$ of matrix $Z$, $\forall i \in \langle m \rangle$, $\forall j \in \langle n \rangle$.

Set

$$C_{m,n} = \{P \mid P \text{ is a complex } m \times n \text{ matrix}\},$$

$$R_{m,n} = \{P \mid P \text{ is a real } m \times n \text{ matrix}\},$$

$$N_{m,n} = \{P \mid P \text{ is a nonnegative } m \times n \text{ matrix}\},$$

$$S_{m,n} = \{P \mid P \text{ is a stochastic } m \times n \text{ matrix}\}$$

(...the stochastic matrices are defined in Definition 1.3),

$$C_n = C_{n,n},$$

$$R_n = R_{n,n},$$

$$N_n = N_{n,n},$$

$$S_n = S_{n,n}.$$

Let $P = (P_{ij}) \in C_{m,n}$. Let $\emptyset \neq U \subseteq \langle m \rangle$ and $\emptyset \neq V \subseteq \langle n \rangle$. Set the matrices

$$P_U = (P_{ij})_{i \in U, j \in \langle n \rangle}, \ P^V = (P_{ij})_{i \in \langle m \rangle, j \in V}, \text{ and } P_U^V = (P_{ij})_{i \in U, j \in V}.$$

**Remark 1.2.** (Basic properties of the operators $(\cdot)_{(\cdot)}$, $(\cdot)^{(\cdot)}$, and $(\cdot)_{(\cdot)}^{(\cdot)}$ on the products of complex matrices.) Let $P_1 \in C_{n_1,n_2}$, $P_2 \in C_{n_2,n_3}$, ..., $P_t \in C_{n_t,n_{t+1}}$, where $t \geq 2$. Let $k \in \langle t-1 \rangle$. Let $\emptyset \neq U \subseteq \langle n_1 \rangle$ and $\emptyset \neq V \subseteq \langle n_{t+1} \rangle$. Obviously, we have

(a)

$$(P_1 P_2 ... P_t)_U = (P_1 P_2 ... P_k)_U P_{k+1} P_{k+2} ... P_t;$$

(b)

$$(P_1 P_2 ... P_t)^V = P_1 P_2 ... P_k (P_{k+1} P_{k+2} ... P_t)^V;$$

(c)

$$(P_1 P_2 ... P_t)_U^V = (P_1 P_2 ... P_k)_U (P_{k+1} P_{k+2} ... P_t)^V;$$

(d)

$$(P_1P_2...P_t)_U^V = \sum_{W\in\Delta} (P_1P_2...P_k)_U^W (P_{k+1}P_{k+2}...P_t)_W^V, \ \forall\Delta \in \operatorname{Par}(\langle n_{k+1}\rangle);$$

(e)

$$(P_1P_2...P_t)_U^V \geq (P_1P_2...P_k)_U^W (P_{k+1}P_{k+2}...P_t)_W^V, \ \forall W, \ \emptyset \neq W \subseteq \langle n_{k+1}\rangle.$$

((c)$\Longrightarrow$(d)$\Longrightarrow$(e))

Let $W$ be a nonempty finite set. Suppose that $W = \{s_1, s_2, ..., s_t\}$. Set

$$(\{i\})_{i\in W} \in \operatorname{Par}(W), \ (\{i\})_{i\in W} = (\{s_1\}, \{s_2\}, ..., \{s_t\}).$$

*E.g.*,

$$(\{i\})_{i\in\langle n\rangle} = (\{1\}, \{2\}, ..., \{n\}).$$

$(\{i\})_{i\in W}$ is the finest partition of $W$ while $(W)$ is the coarsest partition of $W$. Let $L \subset W$. Let $K = L^c$, $L^c$ is the complement of $L$ — so, $\emptyset \neq K \subseteq W$. Suppose that $K = \{s_{j_1}, s_{j_2}, ..., s_{j_u}\}$ $(1 \leq u \leq t)$. Set

$$(\{i\}, L)_{i\in K} \in \operatorname{Par}(W), \ (\{i\}, L)_{i\in K} = \begin{cases} (\{i\})_{i\in W} & \text{if } |L| = 0, \\ (\{s_{j_1}\}, \{s_{j_2}\}, ..., \{s_{j_u}\}, L) & \text{if } |L| \geq 1 \end{cases}$$

($|\cdot|$ is the cardinality). Obviously, $(\{i\}, L)_{i\in K} = (\{i\})_{i\in W}$ if $|L| = 1$. So,

$$(\{i\}, L)_{i\in K} = (\{i\})_{i\in W} \ \text{ if } |L| \leq 1.$$

Further, we suppose that $t \geq 2$. Let $(L_1, L_2, ..., L_m) \in$Par$(W)$, where $m \geq 2$ $(L_1, L_2, ..., L_m \neq \emptyset$; $m \geq 2 \Longrightarrow t \geq 2)$. Suppose that $L_1 = \{s_{l_1}, s_{l_2}, ..., s_{l_v}\}$ $(v \geq 1)$. Set

$$(\{i\}, L_2, ..., L_m)_{i\in L_1} \in \operatorname{Par}(W),$$

$$(\{i\}, L_2, ..., L_m)_{i\in L_1} = (\{s_{l_1}\}, \{s_{l_2}\}, ..., \{s_{l_v}\}, L_2, ..., L_m).$$

*E.g.*,

$$(\{i\}, \{4,5\}, \{6,7,8\})_{i\in\langle 3\rangle} = (\{1\}, \{2\}, \{3\}, \{4,5\}, \{6,7,8\}).$$

$(\{i\}, L)_{i\in K}$ is the finest partition of $W$ when $|L| \leq 1$; $(\{i\}, L)_{i\in K}$ is the finest partition (of $W$) among the partitions of $W$ which contain $L$ when $|L| > 1$ while $(\{i\}, L_2, ..., L_m)_{i\in L_1}$ is the finest partition (of $W$) among the partitions of $W$ which contain $L_2, ..., L_m$. Obviously, $(\{i\}, L_2, ..., L_m)_{i\in L_1} = (\{i\})_{i\in W}$ when $|L_2| = ... = |L_m| = 1$. Obviously, the partition $(\{i\}, L)_{i\in K}$ when $|L| \geq 1$ is a special one of $(\{i\}, L_2, ..., L_m)_{i\in L_1}$.

**Definition 1.3.** *Let $P \in C_{m,n}$. We say that $P$ is a generalized stochastic (complex) matrix if* $\exists a \in \mathbb{C}$ $(a = 0$ *or* $a \neq 0)$ *such that*

$$\sum_{j \in \langle n \rangle} P_{ij} = a, \ \forall i \in \langle m \rangle .$$

(We have generalized stochastic complex matrices, generalized stochastic real matrices, generalized stochastic nonnegative matrices... If $P \in N_{m,n}$, then $P$ is a generalized stochastic (nonnegative) matrix if and only if $\exists a \geq 0$, $\exists Q \in S_{m,n}$ such that $P = aQ$ — see, *e.g.*, [17, Definition 1.2].) The generalized stochastic matrices with row sums equal to 1 are called *stochastic complex* when the matrices are complex, *stochastic real* when the matrices are real, *stochastic nonnegative* or, for short, *stochastic* when the matrices are nonnegative...

**Definition 1.4.** (A generalization of Definition 1.3 from [17].)
*Let $P \in C_{m,n}$. Let $\Delta \in$*Par$(\langle m \rangle)$ *and $\Sigma \in$*Par$(\langle n \rangle)$*. We say that $P$ is a* $[\Delta]$*-stable matrix on $\Sigma$ if $P_K^L$ is a generalized stochastic matrix,* $\forall K \in \Delta$, $\forall L \in \Sigma$. *In particular, a* $[\Delta]$*-stable matrix on* $(\{j\})_{j \in \langle n \rangle}$ *is called* $[\Delta]$*-stable for short.*

**Definition 1.5.**(A generalization of Definition 1.4 from [17].)
*Let $P \in C_{m,n}$.* Let $\Delta \in$Par$(\langle m \rangle)$ *and* $\Sigma \in$Par$(\langle n \rangle)$. *We say that $P$ is a $\Delta$-stable matrix on $\Sigma$ if $\Delta$ is the least fine partition for which $P$ is a* $[\Delta]$*-stable matrix on $\Sigma$. In particular, a $\Delta$-stable matrix on* $(\{j\})_{j \in \langle n \rangle}$ *is called $\Delta$-stable while a* $(\langle m \rangle)$*-stable matrix on $\Sigma$ is called stable on $\Sigma$ for short. A stable matrix on* $(\{j\})_{j \in \langle n \rangle}$ *is called stable for short.*

For the $[\Delta]$- and $\Delta$-stable matrices on $\Sigma$, $\Delta \in$Par$(\langle m \rangle)$, $\Sigma \in$Par$(\langle n \rangle)$, we will use the generic name "generalized stable (complex) matrices" — we have generalized stable complex matrices, generalized stable real matrices, generalized stable nonnegative matrices, generalized stable stochastic matrices...

The stable matrices have a central role in the finite Markov chain theory, in the DeGroot model theory (see Section 2), in the theory of DeGroot model on distributed systems (see Section 3)... The stable matrices on $\Sigma$ are important for the finite Markov chains (see, *e.g.*, [17, p. 389]), for the partial consensuses (see, in Section 2, Theorem 2.5; see also Example 2.8), for the partial distributed consensuses (see, in Section 3, Remark 3.2(b))...

**Example 1.6.** Let

$$P = \begin{pmatrix} 2 & 3 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 4 & 0 & \frac{1}{2} & \frac{1}{2} \\ 4 & 6 & 0 & 0 & 7 \\ 9 & 1 & 1 & 2 & 4 \end{pmatrix}, \ Q = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

$$T=\begin{pmatrix} 3 & -\frac{1}{4} & \frac{1}{4} & \frac{2}{4} \\ 3 & -\frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}, \text{ and } Z=\begin{pmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{pmatrix}.$$

$P$ is $[(\{1,2\},\{3,4\})]$-stable on $(\{1,2\},\{3,4,5\})$ because $P_K^L$ is a generalized stochastic matrix, $\forall K\in(\{1,2\},\{3,4\})$, $\forall L\in(\{1,2\},\{3,4,5\})$ — indeed,

$$P_{\{1,2\}}^{\{1,2\}}=\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix},$$

and it is a generalized stochastic matrix because $2+3=1+4$,

$$P_{\{1,2\}}^{\{3,4,5\}}=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and it is a generalized stochastic matrix, etc. Moreover, $P$ is $[\Delta]$-stable on $(\{1,2\},\{3,4,5\})$, $\forall\Delta\preceq(\{1,2\},\{3,4\})$. Moreover, $P$ is $(\{1,2\},\{3,4\})$-stable on $(\{1,2\},\{3,4,5\})$. $Q$ is $[(\{1,2\},\{3,4\})]$-stable

$$(Q_{\{1,2\}}^{\{1\}}=\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ Q_{\{1,2\}}^{\{2\}}\equiv Q_{\{1,2\}}^{\{3\}}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\cdots$$

— $\equiv$ is the identical sign). Moreover, it is $[\Delta]$-stable, $\forall\Delta\preceq(\{1,2\},\{3,4\})$. Moreover, it is $(\{1,2\},\{3,4\})$-stable. $T$ is stable on $(\{1\},\{2\},\{3,4\})$ — it follows that $T^{\langle 2\rangle}$ is stable (the rows of $T^{\langle 2\rangle}$ are identical — any stable matrix has the rows identically). $Z$ is stable (the rows of $Z$ are identical).

Let $\Delta\in\mathrm{Par}(\langle m\rangle)$ and $\Sigma\in\mathrm{Par}(\langle n\rangle)$. Set (see [17] for $G_{\Delta,\Sigma}$ and [18] for $\overline{G}_{\Delta,\Sigma}$)

$$G_{\Delta,\Sigma}=G_{\Delta,\Sigma}(m,n)=\{P\mid P\in S_{m,n}\text{ and }P\text{ is a }[\Delta]\text{-stable matrix on }\Sigma\},$$

$$\overline{G}_{\Delta,\Sigma}=\overline{G}_{\Delta,\Sigma}(m,n)=\{P\mid P\in N_{m,n}\text{ and }P\text{ is a }[\Delta]\text{-stable matrix on }\Sigma\},$$

$$\widetilde{G}_{\Delta,\Sigma}=\widetilde{G}_{\Delta,\Sigma}(m,n)=\{P\mid P\in R_{m,n}\text{ and }P\text{ is a }[\Delta]\text{-stable matrix on }\Sigma\},$$

and

$$\widehat{G}_{\Delta,\Sigma}=\widehat{G}_{\Delta,\Sigma}(m,n)=\{P\mid P\in C_{m,n}\text{ and }P\text{ is a }[\Delta]\text{-stable matrix on }\Sigma\}$$

($G_{\Delta,\Sigma}$ is a set of generalized stable stochastic matrices, $\overline{G}_{\Delta,\Sigma}$ is a set of generalized stable nonnegative matrices...).

When we study or even when we construct products of complex matrices (in particular, products of stochastic matrices) using $G_{\Delta,\Sigma}$, $\overline{G}_{\Delta,\Sigma}$, $\widetilde{G}_{\Delta,\Sigma}$, or $\widehat{G}_{\Delta,\Sigma}$, we shall refer this as the $G$ *method*. $G$ comes from the verb *to group* and its derivatives.

The $G$ method can more be extended considering other sets of matrices besides $S_{m,n}$, $N_{m,n}$, $R_{m,n}$, and $C_{m,n}$, then, if necessary, generalizing Definition 1.3...

The $G$ method has serious applications, see, *e.g.*, [20]-[21].

Since $G_{\Delta,\Sigma} \subseteq \overline{G}_{\Delta,\Sigma} \subseteq \widetilde{G}_{\Delta,\Sigma} \subseteq \widehat{G}_{\Delta,\Sigma}$ ($\Delta \in \mathrm{Par}(\langle m\rangle)$, $\Sigma \in \mathrm{Par}(\langle n\rangle)$), any property which holds for $\widehat{G}_{\Delta,\Sigma}$ also holds for $G_{\Delta,\Sigma}$, $\overline{G}_{\Delta,\Sigma}$, and $\widetilde{G}_{\Delta,\Sigma}$, any property which holds for $\widetilde{G}_{\Delta,\Sigma}$ also holds for $G_{\Delta,\Sigma}$ and $\overline{G}_{\Delta,\Sigma}$... Several properties of $G_{\Delta,\Sigma}$ were given in Remark 2.1 from [17] — some of these properties can be extended for $\overline{G}_{\Delta,\Sigma}$, for $\widetilde{G}_{\Delta,\Sigma}$, or for $\widehat{G}_{\Delta,\Sigma}$. Below we give a few properties of $G_{\Delta,\Sigma}$, $\overline{G}_{\Delta,\Sigma}$, $\widetilde{G}_{\Delta,\Sigma}$, or $\widehat{G}_{\Delta,\Sigma}$.

**Remark 1.7.** (a)

$$S_{m,n} = G_{(\langle m\rangle),(\langle n\rangle)} = G_{(\{i\})_{i\in\langle m\rangle},\Sigma},\ \forall m,n \geq 1,\ \forall \Sigma \in \mathrm{Par}\left(\langle n\rangle\right)$$

($(\langle m\rangle)$ is the improper (degenerate) partition of $\langle m\rangle$...; $\Sigma$ can be equal to $(\{j\})_{j\in\langle n\rangle}$ (so, $S_{m,n} = G_{(\{i\})_{i\in\langle m\rangle},(\{j\})_{j\in\langle n\rangle}}$), or, more generally, to $(\{j\}, K^c)_{j\in K}$, $\emptyset \neq K \subseteq \langle n\rangle$, or to...).

(b)

$$N_{m,n} \supset \overline{G}_{(\langle m\rangle),(\langle n\rangle)},\ R_{m,n} \supset \widetilde{G}_{(\langle m\rangle),(\langle n\rangle)},\ C_{m,n} \supset \widehat{G}_{(\langle m\rangle),(\langle n\rangle)},\ \forall m \geq 2,\ \forall n \geq 1,$$

and $\overline{G}_{(\langle m\rangle),(\langle n\rangle)}$ is equal to the set of generalized stochastic nonnegative $m \times n$ matrices, $\widetilde{G}_{(\langle m\rangle),(\langle n\rangle)}$ is equal to the set of generalized stochastic real $m \times n$ matrices, and $\widehat{G}_{(\langle m\rangle),(\langle n\rangle)}$ is equal to the set of generalized stochastic complex $m \times n$ matrices;

$$N_{m,n} = \overline{G}_{(\{i\})_{i\in\langle m\rangle},\Sigma},\ R_{m,n} = \widetilde{G}_{(\{i\})_{i\in\langle m\rangle},\Sigma},\ C_{m,n} = \widehat{G}_{(\{i\})_{i\in\langle m\rangle},\Sigma},$$

$\forall m \geq 1$, $\forall n \geq 1$, $\forall \Sigma \in \mathrm{Par}(\langle n\rangle)$.

(c) The stable complex $m \times n$ matrices on $\Sigma$ are generalized stochastic, $\forall \Sigma \in \mathrm{Par}(\langle n\rangle)$ ($\overline{G}_{(\langle m\rangle),\Sigma}, \widetilde{G}_{(\langle m\rangle),\Sigma}$, and $\widehat{G}_{(\langle m\rangle),\Sigma}$ are included in the set of generalized stochastic nonnegative $m \times n$ matrices, that of generalized stochastic real $m\times n$ matrices, and that of generalized stochastic complex $m\times n$ matrices, respectively).

(d)

$$\overline{G}_{(\langle m\rangle),(\{j\},L_2,L_3,...,L_v)_{j\in L_1}} \subseteq \overline{G}_{(\langle m\rangle),\left(\{j\},\bigcup\limits_{b=2}^{v} L_b\right)_{j\in L_1}},$$

$\forall m$, $n \geq 1$, $\forall (L_1,L_2,...,L_v) \in \mathrm{Par}(\langle n\rangle)$, $v \geq 2$ (obviously, $(\{j\}, L_2, L_3, ..., L_v)_{j\in L_1} \in \mathrm{Par}(\langle n\rangle)$). Similarly for $\widetilde{G}_{(\langle m\rangle),(\{j\},L_2,L_3,...,L_v)_{j\in L_1}}$ and for $\widehat{G}_{(\langle m\rangle),(\{j\},L_2,L_3,...,L_v)_{j\in L_1}}$.

Let $P \in \widehat{G}_{\Delta,\Sigma}$. Then (see Definition 1.4) $\forall K \in \Delta$, $\forall L \in \Sigma$, $\exists a_{K,L} = a_{K,L}(P) \in \mathbb{C}$ such that

$$\sum_{j \in L} P_{ij} = a_{K,L}, \forall i \in K.$$

Set the matrix (for the case when $P \in \overline{G}_{\Delta,\Sigma}$, see, *e.g.*, also [18])

$$P^{-+} = \left(P^{-+}_{KL}\right)_{K \in \Delta, L \in \Sigma}, \; P^{-+}_{KL} = a_{K,L}, \; \forall K \in \Delta, \; \forall L \in \Sigma.$$

We call the matrix $P^{-+}$ the $(\Delta, \Sigma)$-*grouped matrix* (*of* $P$), or, for short, if no confusion can arise, the *grouped matrix* (*of* $P$) — in [17] this matrix was differently called... The labels of rows and columns of $P^{-+}$ are $K$, $K \in \Delta$, and $L$, $L \in \Sigma$, respectively; $P^{-+}_{KL}$, $K \in \Delta$, $L \in \Sigma$, are the entries of matrix $P^{-+}$; $P^{-+}_{KL}$ is the entry $(K, L)$ of matrix $P^{-+}$, $\forall K \in \Delta$, $\forall L \in \Sigma$. If confusion can arise, we write $P^{-+(\Delta,\Sigma)}$ instead of $P^{-+}$.

**Example 1.8.** Consider the matrices $P$, $Q$, $T$, and $Z$ from Example 1.6. We have

$$P^{-+} = \begin{pmatrix} 5 & 1 \\ 10 & 7 \end{pmatrix} \; (\Delta = (\{1,2\},\{3,4\}), \; \Sigma = (\{1,2\},\{3,4,5\})),$$

$$Q^{-+} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix} \; (\Delta = (\{1,2\},\{3,4\}), \; \Sigma = (\{j\})_{j \in \langle 3 \rangle}),$$

$$T^{-+} = \begin{pmatrix} 3 & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}, \; Z^{-+} = \begin{pmatrix} \frac{1}{4} & -\frac{3}{4} \end{pmatrix}.$$

$T^{-+}$ and $Z^{-+}$ have just one row because $T$ and $Z$ are stable on $(\{1\},\{2\},\{3,4\})$ and stable, respectively. Moreover, the row of $\left(T^{\langle 2 \rangle}\right)^{-+} = \left(T^{\langle 2 \rangle}\right)^{-+((\langle 2 \rangle),(\{1\},\{2\}))}$ is identical with any row of $T^{\langle 2 \rangle}$ and the row of $Z^{-+}$ is identical with any row of $Z$. Generally speaking, letting $A$ and $B$, $A$, $B \in C_{m,n}$, be a stable matrix on $(\{j\}, K^c)_{j \in K}$, $\emptyset \neq K \subseteq \langle n \rangle$, $|K^c| > 1$, and a stable matrix, respectively, $\left(A^K\right)^{-+} = \left(A^K\right)^{-+\left((\langle m \rangle),(\{j\})_{j \in K}\right)}$ has just one row and this is identical with any row of $A^K$ and $B^{-+} = B^{-+\left((\langle m \rangle),(\{j\})_{j \in \langle n \rangle}\right)}$ has just one row and this is identical with any row of $B$ — do not forget! (see Theorem 1.10...)

Below we give a fundamental result.

**Theorem 1.9.** (i) (a generalization of Theorem 1.5 from [18]) *Let* $P_1 \in \widehat{G}_{\Delta_1,\Delta_2} \subseteq C_{n_1,n_2}$ *and* $P_2 \in \widehat{G}_{\Delta_2,\Delta_3} \subseteq C_{n_2,n_3}$. *Then*

$$P_1 P_2 \in \widehat{G}_{\Delta_1,\Delta_3} \subseteq C_{n_1,n_3} \quad \text{and} \quad (P_1 P_2)^{-+} = P_1^{-+} P_2^{-+}.$$

(ii) (a generalization of (i)) *Let* $P_1 \in \widehat{G}_{\Delta_1,\Delta_2} \subseteq C_{n_1,n_2}$, $P_2 \in \widehat{G}_{\Delta_2,\Delta_3} \subseteq C_{n_2,n_3}$, ..., $P_t \in \widehat{G}_{\Delta_t,\Delta_{t+1}} \subseteq C_{n_t,n_{t+1}}$, *where* $t \geq 2$. *Then*

$$P_1 P_2 ... P_t \in \widehat{G}_{\Delta_1,\Delta_{t+1}} \subseteq C_{n_1,n_{t+1}} \quad \text{and} \quad (P_1 P_2 ... P_t)^{-+} = P_1^{-+} P_2^{-+} ... P_t^{-+}.$$

*Proof.* (i) Similar to the proof of Theorem 1.5 from [18]. Another proof, a simple one too, of the first part of (i) is as follows. Let $U \in \Delta_1$ and $V \in \Delta_3$. By Remark 1.2(d) we have

$$(P_1P_2)_U^V = \sum_{W \in \Delta_2} (P_1)_U^W (P_2)_W^V .$$

Since $(P_1)_U^W$ and $(P_2)_W^V$ are generalized stochastic matrices, $\forall W \in \Delta_2$, it follows that

$$(P_1)_U^W (P_2)_W^V \text{ and } \sum_{W \in \Delta_2} (P_1)_U^W (P_2)_W^V$$

are generalized stochastic matrices. So, $(P_1P_2)_U^V$ is a generalized stochastic matrix. Therefore, $P_1P_2 \in \widehat{G}_{\Delta_1,\Delta_3}$.

(ii) Induction. ∎

Recall that in this article, a vector is a row vector. Set $e = e(n) = (1, 1, ..., 1) \in \mathbb{R}^n$ ($e \in \mathbb{C}^n$ too), $\forall n \geq 1$; $e'$ is its transpose.

Below we give the best result of this section.

**Theorem 1.10.** (An extension of Theorem 1.6 from [18].) *Let* $P_1 \in \widehat{G}_{\Delta_1,\Delta_2} \subseteq C_{n_1,n_2}$, $P_2 \in \widehat{G}_{\Delta_2,\Delta_3} \subseteq C_{n_2,n_3}$, ..., $P_t \in \widehat{G}_{\Delta_t,\Delta_{t+1}} \subseteq C_{n_t,n_{t+1}}$, *where* $t \geq 2$. *Suppose that*

$$\Delta_1 = (\langle n_1 \rangle) \quad \textit{and } \Delta_{t+1} = (\{j\}, K^c)_{j \in K}$$

($\emptyset \neq K \subseteq \langle n_{t+1} \rangle$; $|K^c| \leq 1$ *or* $|K^c| > 1$). *Then*

$$(P_1P_2...P_t)^K$$

$$(\textit{equivalently, } P_1P_2...P_{t-1}P_t^K \textit{ (see Remark } 1.2\text{(b)))}$$

*is a stable matrix (i.e., a matrix with identical rows, see Definition* 1.5 *and Example* 1.6; $(P_1P_2...P_t)^K$ *is a submatrix of* $P_1P_2...P_t$, *see the definition of operator* $(\cdot)^{(\cdot)}$ *again; if* $K = \langle n_{t+1} \rangle$, *then* $(P_1P_2...P_t)^K = P_1P_2...P_t$, and $P_1P_2...P_t$ *is a stable matrix*),

$$(P_1P_2...P_t)_{\{i\}}^K = \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j \in K}\{\{j\}\}} =$$

$$= P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+}, \ \forall i \in \langle n_1 \rangle,$$

*where*

$$\left(P_t^K\right)^{-+} = \left(P_t^K\right)^{-+\left(\Delta_t, (\{j\})_{j \in K}\right)}$$

$((P_1P_2...P_t)^K_{\{i\}}$ *is the row* $i$ *of matrix* $(P_1P_2...P_t)^K$; $\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}}$ *is a submatrix of* $P_1^{-+}P_2^{-+}...P_t^{-+}$ *(see the definition of operator* $(\cdot)^{(\cdot)}$*);* $P_b^{-+} = P_b^{-+(\Delta_b,\Delta_{b+1})}, \forall b \in \langle t \rangle$*; we used* $\{\{j\}\}$ *because the labels of columns of* $P_1^{-+}P_2^{-+}...P_t^{-+}$ *are, here,* $\{j\}$ *,* $j \in \langle n_{t+1} \rangle$ *, when* $|K^c| \leq 1$ *and* $\{j\}$ *,* $j \in K$*, and* $K^c$ *when* $|K^c| > 1$ *(see the definition of operator* $(\cdot)^{-+}$ *again)), and*

$$(P_1P_2...P_t)^K = e' \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}} = e'P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+},$$

*where* $e = e(n_1)$ *(if* $K = \langle n_{t+1} \rangle$ *, then*

$$(P_1P_2...P_t)_{\{i\}} = P_1^{-+}P_2^{-+}...P_t^{-+},\ \forall i \in \langle n_1 \rangle,$$

*and*

$$P_1P_2...P_t = e'P_1^{-+}P_2^{-+}...P_t^{-+}).$$

*Proof.* Since $\Delta_1 = (\langle n_1 \rangle)$ and $\Delta_{t+1} = (\{j\}, K^c)_{j\in K}$, by Theorem 1.9(ii) and Definition 1.5, $P_1P_2...P_t$ is a stable matrix on $(\{j\}, K^c)_{j\in K}$ ($|K^c| \leq 1$ or $|K^c| > 1$). It follows that $(P_1P_2...P_t)^K$ is a stable matrix and, consequently,

$$\left((P_1P_2...P_t)^K\right)^{-+\left((\langle n_1\rangle),(\{j\})_{j\in K}\right)}$$

has only one row — the label of this row is $\langle n_1 \rangle$ (see the definition of operator $(\cdot)^{-+}$); further, we have

$$(P_1P_2...P_t)^K_{\{i\}} = \left(P_1P_2...P_{t-1}P_t^K\right)_{\{i\}} =$$

(due to the partitions $(\langle n_1 \rangle)$ and $(\{j\})_{j\in K}$ ... — see Example 1.8 again)

$$= \left(P_1P_2...P_{t-1}P_t^K\right)^{-+\left((\langle n_1\rangle),(\{j\})_{j\in K}\right)} =$$

$$= P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+},\ \forall i \in \langle n_1 \rangle.$$

On the other hand, it is easy to see (see also Example 1.8) that

$$(P_1P_2...P_t)^K_{\{i\}} = \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}},\ \forall i \in \langle n_1 \rangle.$$

Finally, we have

$$(P_1P_2...P_t)^K = e' \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}} = e'P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+}.$$

∎

**Remark 1.11.** On Theorem 1.10, we consider 3 things.

(a) $\Delta_{t+1} = (\{j\})_{j\in\langle n_{t+1}\rangle}$ both when $|K^c| = 0$ and when $|K^c| = 1$ — practically speaking, we use "$|K^c| = 0$" because, in this case, $K = \langle n_{t+1}\rangle$.

(b) We have, in fact,

$$(P_1P_2...P_t)^K_{\{i\}} \equiv \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}}, \ \forall i \in \langle n_1\rangle,$$

because the row matrices

$$(P_1P_2...P_t)^K_{\{i\}} \text{ and } \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}}$$

have different labels for rows and, separately, for columns, $\forall i \in \langle n_1\rangle$, but abusively, for simplification, we used " $=$ " instead of " $\equiv$ " ($\equiv$ is the identical sign; since the one-to-one correspondence (the bijective function) is column $j \longmapsto$ column (with label) $\{j\}$, $\forall j \in K$, obviously, $\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}}$ must have the property: the column $\{j_1\}$ is before the column $\{j_2\}$ if $j_1 < j_2$, $\forall j_1, j_2 \in K$). Also,

$$(P_1P_2...P_t)^K_{\{i\}} \equiv P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+}, \ \forall i \in \langle n_1\rangle,$$

but, due to the notion of equal matrices and notation used for the equal matrices,

$$\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}} = P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+}.$$

Also,

$$(P_1P_2...P_t)^K \equiv e'\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}} ...$$

(c) Theorem 1.10 leads to another one if we replace $\Delta_{t+1} = (\{j\}, K^c)_{j\in K}$ with $\Delta_{t+1} = (\{j\}, L_2, L_3, ..., L_v)_{j\in L_1}$, where $(L_1, L_2, L_3, ..., L_v) \in$Par$(\langle n_{t+1}\rangle)$ and $v \geq 2$. But, by Remark 1.7(d), $P_1P_2...P_t$ is stable on $(\{j\}, L_2, L_3, ..., L_v)_{j\in L_1}$ $\Longrightarrow P_1P_2...P_t$ is stable on $\left(\{j\}, \bigcup\limits_{b=2}^{v} L_b\right)_{j\in L_1}$. So, nothing new.

**Example 1.12.** Let

$$P_1 = \begin{pmatrix} \frac{2}{4} & 0 & \frac{2}{4} & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{2}{4} & \frac{2}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{pmatrix}, \ P_2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{2}{4} \\ \frac{2}{4} & 0 & 0 & \frac{2}{4} \\ \frac{2}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{2}{4} & \frac{2}{4} & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}, \text{ and } P_3' = \begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{2}{4} \\ 0 & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{pmatrix}.$$

We have $P_1 \in G_{(\langle 4\rangle),(\langle 4\rangle)}$, $P_2 \in G_{(\langle 4\rangle),(\{1,2\},\{3,4\})}$, $P_3 \in G_{(\{1,2\},\{3,4\}),(\{j\})_{j\in\langle 4\rangle}}$, and $P_3' \in G_{(\{1,2\},\{3,4\}),(\{1\},\{2\},\{3,4\})}$. ($P_1 \notin G_{(\langle 4\rangle),\Sigma}$, $\forall \Sigma \in \mathrm{Par}(\langle 4\rangle)$, $\Sigma \neq (\langle 4\rangle)$ — $P_1$ was deliberately chosen with this property.) So, by Theorem 1.10, $P_1P_2P_3$ is a stable matrix and

$$P_1P_2P_3 = e'P_1^{-+}P_2^{-+}P_3^{-+} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (1) \begin{pmatrix} \frac{2}{4} & \frac{2}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{2}{16} & \frac{2}{16} & \frac{10}{16} & \frac{2}{16} \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{5}{8} & \frac{1}{8} \end{pmatrix}$$

while $(P_1P_2P_3')^{\{1,2\}}$ is a stable matrix and

$$\left(P_1P_2P_3'\right)^{\{1,2\}} = e'\left(P_1^{-+}P_2^{-+}P_3'^{-+}\right)^{\{\{1\},\{2\}\}} = e'P_1^{-+}P_2^{-+}\left(P_3'^{-+}\right)^{\{\{1\},\{2\}\}} =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (1) \begin{pmatrix} \frac{2}{4} & \frac{2}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{2}{16} & \frac{2}{16} \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}.$$

These results can also be obtained by direct computation. $P_2P_3$ and $(P_2P_3')^{\{1,2\}}$ are also stable matrices.

To easy recognize when Theorem 1.10 could or cannot be used/applied, see (i) and (ii) in the next theorem.

**Theorem 1.13.** *Suppose that the conditions of Theorem* 1.10 *hold.*

(i) *If* $n_2, n_3, ..., n_{t+1} \geq 2$ ($n_1 \geq 1$), *then* $\exists u \in \langle t\rangle$ *such that* $P_u$ *is stable on* $\Delta_{u+1}$, *and* $\Delta_{u+1} \neq (\langle n_{u+1}\rangle)$ ($P_u$ *can be "broken" into at least two generalized stochastic matrices with* $n_u$ *rows*).

(ii) *If* $n_1, n_2, ..., n_t \geq 2$ ($n_{t+1} \geq 1$), *then* $\exists w \in \langle t-1\rangle$ ($t \geq 2$) *such that* $\Delta_w \neq (\{i\})_{i\in\langle n_w\rangle}$ *and* $(P_w)_U$ *is a matrix with identical rows*, $\forall U \in \Delta_w$, $|U| \geq 2$ (*as a result,* $P_w$ *has at least two identical rows*) *or* $(P_t)_U^K$ *is a matrix with* *identical rows,* $\forall U \in \Delta_t$*,* $|U| \geq 2$ *(as a result,* $P_t^K$ *has at least two identical rows (*$P_t^K = P_t$ *when* $K = \langle n_{t+1} \rangle$*)).*

(iii) $P_1$*,* $P_1P_2$*, ...,* $P_1P_2...P_t$ *are stable on* $\Delta_2$*,* $\Delta_3$*, ...,* $\Delta_{t+1}$ *(*$\Delta_{t+1} = (\{j\}, K^c)_{j \in K}$*), respectively, and, as a result, they are generalized stochastic matrices.*

(iv) $P_t$*,* $P_{t-1}P_t$*, ...,* $P_1P_2...P_t$ *are* $[\Delta_t]$*-,* $[\Delta_{t-1}]$*-, ...,* $[\Delta_1]$*-stable on* $(\{j\}, K^c)_{j \in K}$*, respectively (*$\Delta_1 = (\langle n_1 \rangle)$*).*

*Proof.* (i) Since $n_2, n_3, ..., n_{t+1} \geq 2$, $\Delta_1 = (\langle n_1 \rangle)$, and $\Delta_{t+1} = (\{j\}, K^c)_{j \in K}$, it follows that $\exists u \in \langle t \rangle$ such that $\Delta_1 = (\langle n_1 \rangle)$, $\Delta_2 = (\langle n_2 \rangle)$, ..., $\Delta_u = (\langle n_u \rangle)$, and $\Delta_{u+1} \neq (\langle n_{u+1} \rangle)$ ("$u = t$" leads to "$\Delta_{t+1} \neq (\langle n_{t+1} \rangle)$"; $n_{t+1} \geq 2$ and $\emptyset \neq K \subseteq \langle n_{t+1} \rangle \Longrightarrow |\Delta_{t+1}| \geq 2$ (equivalently, $\Delta_{t+1} \neq (\langle n_{t+1} \rangle)$); so, no problem). It follows that $P_u$ is stable on $\Delta_{u+1}$, and $\Delta_{u+1} \neq (\langle n_{u+1} \rangle)$.

(ii) *Case* 1. $\Delta_t \neq (\{i\})_{i \in \langle n_t \rangle}$ — this case is possible because $n_t \geq 2$. In this case, $\exists V \in \Delta_t$ with $|V| \geq 2$. Since $P_t \in \widehat{G}_{\Delta_t, \Delta_{t+1}}$ and $\Delta_{t+1} = (\{j\}, K^c)_{j \in K}$, it follows that $(P_t)_U^K$ is a (complex) matrix with identical rows, $\forall U \in \Delta_t$, $|U| \geq 2$.

*Case* 2. $\Delta_t = (\{i\})_{i \in \langle n_t \rangle}$. Since $n_1, n_2, ..., n_{t-1} \geq 2$ (here, $n_{t-1}$, not $n_t$), $\Delta_1 = (\langle n_1 \rangle)$, and $\Delta_t = (\{i\})_{i \in \langle n_t \rangle}$, it follows that $\exists w \in \langle t-1 \rangle$ ($t \geq 2$) such that $\Delta_t = (\{i\})_{i \in \langle n_t \rangle}$, $\Delta_{t-1} = (\{i\})_{i \in \langle n_{t-1} \rangle}$, ..., $\Delta_{w+1} = (\{i\})_{i \in \langle n_{w+1} \rangle}$, and $\Delta_w \neq (\{i\})_{i \in \langle n_w \rangle}$ ($n_w \geq 2$, so, "$\Delta_w \neq (\{i\})_{i \in \langle n_w \rangle}$" is possible). Since $P_w \in \widehat{G}_{\Delta_w, \Delta_{w+1}}$, it follows that $(P_w)_U$ is a matrix with identical rows, $\forall U \in \Delta_w$, $|U| \geq 2$ ($\Delta_w \neq (\{i\})_{i \in \langle n_w \rangle} \Longrightarrow \exists V \in \Delta_w$ with $|V| \geq 2$).

(iii) and (iv) Theorem 1.9. ∎

Theorem 1.10 was illustrated in Example 1.12. To illustrate Theorem 1.13(i)-(ii), considering Example 1.12, the matrix $P_1$ cannot be "broken" into at least two generalized stochastic matrices with 4 rows, the matrix $P_2$ can be "broken" into two generalized stochastic (nonnegative) matrices with 4 rows, namely, $P_2^{\langle 2 \rangle}$ and $P_2^{\{3,4\}}$,

$$P_2^{\langle 2 \rangle} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{2}{4} & 0 \\ \frac{2}{4} & 0 \\ 0 & \frac{2}{4} \end{pmatrix} \text{ and } P_2^{\{3,4\}} = \begin{pmatrix} 0 & \frac{2}{4} \\ 0 & \frac{2}{4} \\ \frac{1}{4} & \frac{1}{4} \\ \frac{2}{4} & 0 \end{pmatrix},$$

the matrix $P_3$ is $[(\langle 2 \rangle, \{3,4\})]$-stable, so, $(P_3)_{\langle 2 \rangle}$ and $(P_3)_{\{3,4\}}$ are stable matrices

$$(\,(P_3)_{\langle 2 \rangle} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \end{pmatrix} ...),$$

and the matrix $P_3^{\langle 2\rangle}$ is $[(\langle 2\rangle, \{3,4\})]$-stable, so, $(P_3)_{\langle 2\rangle}^{\langle 2\rangle}$ and $(P_3)_{\{3,4\}}^{\langle 2\rangle}$ are stable matrices.

Below we consider a similarity relation, and give a result about it — the relation and result lead, under certain conditions, to a subset/subgroup property of the DeGroot submodel from Section 2, see Example 2.11.

**Definition 1.14.** (A generalization of Definition 2.1 from [19].)
*Let $P, Q \in \widehat{G}_{\Delta,\Sigma} \subseteq C_{m,n}$. We say that $P$ is similar to $Q$ (with respect to $\Delta$ and $\Sigma$) if*

$$P^{-+} = Q^{-+}.$$

*Set $P \backsim Q$ when $P$ is similar to $Q$. Obviously, $\backsim$ is an equivalence relation on $\widehat{G}_{\Delta,\Sigma}$.*

**Example 1.15.** (For another example, see Example 1.1 in [21].) Let

$$P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \ Q = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{2}{4} & 0 & \frac{2}{4} & 0 \\ 0 & \frac{2}{4} & \frac{2}{4} & 0 \end{pmatrix}, \text{ and } T = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{2}{4} & \frac{2}{4} & 0 \\ \frac{2}{4} & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

We have $P, Q, R \in G_{\Delta,\Sigma} \subseteq S_4$, where $\Delta = \Sigma = (\{1,2\}, \{3,4\})$, and

$$P^{-+} = Q^{-+} = T^{-+} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{2}{4} & \frac{2}{4} \end{pmatrix}.$$

So, $P \backsim Q \backsim T$. Note that: 1) any matrix which is similar to $P$ with respect to $\Delta$ and $\Sigma$ has at least 8 positive entries; 2) $Q$ has the smallest possible number of positive entries, *i.e.*, 8, while $T$ has not — the matrices with a small number of positive entries are important for algorithms (for sampling...).

The next result says, among other things, that under certain conditions a product of matrices can be replaced with another product of matrices.

**Theorem 1.16.** (A generalization of Theorem 2.3 from [19].)
*Let $P_1, U_1 \in \widehat{G}_{\Delta_1,\Delta_2} \subseteq C_{n_1,n_2}$, $P_2, U_2 \in \widehat{G}_{\Delta_2,\Delta_3} \subseteq C_{n_2,n_3}$, ..., $P_t, U_t \in \widehat{G}_{\Delta_t,\Delta_{t+1}} \subseteq C_{n_t,n_{t+1}}$. Suppose that*

$$P_1 \backsim U_1,\ P_2 \backsim U_2,\ ..., P_t \backsim U_t.$$

*Then*

$$P_1P_2...P_t \backsim U_1U_2...U_t.$$

*If, moreover, $\Delta_1 = (\langle n_1\rangle)$ and $\Delta_{t+1} = (\{j\}, K^c)_{j\in K}$, where $\emptyset \neq K \subseteq \langle n_{t+1}\rangle$, then*

$$(P_1P_2...P_t)^K = (U_1U_2...U_t)^K$$

— *equivalently*, $P_1P_2...P_{t-1}P_t^K = U_1U_2...U_{t-1}U_t^K$

(*therefore, when* $\Delta_1 = (\langle n_1\rangle)$ *and* $\Delta_{t+1} = (\{j\}, K^c)_{j\in K}$, *a product of* $n$ *representatives, the first of an equivalence class included in* $\widehat{G}_{\Delta_1,\Delta_2}$, *the second of an equivalence class included in* $\widehat{G}_{\Delta_2,\Delta_3}$, ..., *the* $t$th *of an equivalence class included in* $\widehat{G}_{\Delta_t,(\{j\})_{j\in K}}$, *does not depend on the choice of representatives — we can work* (*this is important*) *with* $U_1U_2...U_{t-1}U_t^K$ *instead of* $P_1P_2...P_{t-1}P_t^K$ *and, conversely, with* $P_1P_2...P_{t-1}P_t^K$ *instead of* $U_1U_2...U_{t-1}U_t^K$; *due to Theorem* 1.10, $P_1P_2...P_{t-1}P_t^K$ *and* $U_1U_2...U_{t-1}U_t^K$ *are stable matrices*; $P_1P_2...P_{t-1}P_t^K = P_1P_2...P_t$ *and* $U_1U_2...U_{t-1}U_t^K = U_1U_2...U_t$ *when* $K = \langle n_{t+1}\rangle$).

*Proof.* The first part is similar to the proof of first part of Theorem 2.3 from [19]. For the second part, we consider two cases.

*Case* 1. $|K^c| \leq 1$. The proof is similar to the proof of second part of Theorem 2.3 from [19].

*Case* 2. $|K^c| > 1$. We have (see Remark 1.2)

$$(P_1P_2...P_t)^K = P_1P_2...P_{t-1}P_t^K$$

and

$$(U_1U_2...U_t)^K = U_1U_2...U_{t-1}U_t^K.$$

Since $P_t, U_t \in \widehat{G}_{\Delta_t,(\{j\},K^c)_{j\in K}}$, we have $P_t^K, U_t^K \in \widehat{G}_{\Delta_t,(\{j\})_{j\in K}}$. To finish the proof, we apply Case 1 to the matrices $P_1$, $P_2$, ..., $P_{t-1}$, $P_t^K$ and $U_1$, $U_2$, ..., $U_{t-1}$, $U_t^K$, and obtain

$$P_1P_2...P_{t-1}P_t^K = U_1U_2...U_{t-1}U_t^K.$$

∎

For examples for Theorem 1.16, see Example 1.2 in [21] and, in this article, Example 2.11 (in Section 2).

**Remark 1.17.** The equivalence relation $\backsim$ is, in particular, an equivalence relation on the set of generalized stochastic complex $m \times n$ matrices. Indeed, by Remark 1.7(b) we have that $\widehat{G}_{(\langle m\rangle),(\langle n\rangle)}$ is equal to the set of generalized stochastic complex $m \times n$ matrices. Since $\backsim$ is an equivalence relation (with respect to $(\langle m\rangle)$ and $(\langle n\rangle)$) on $\widehat{G}_{(\langle m\rangle),(\langle n\rangle)}$, it follows that it is an equivalence relation (with respect to $(\langle m\rangle)$ and $(\langle n\rangle)$ — see Definition 1.14 again) on the set of generalized stochastic complex $m \times n$ matrices. Similarly, $\backsim$ is an equivalence relation on the set of generalized stochastic real $m \times n$ matrices and on that of generalized stochastic nonnegative $m \times n$ matrices.

## 2. FINITE-TIME CONSENSUS FOR DEGROOT MODEL

In this section, we consider the (homogeneous and nonhomogeneous) DeGroot model and the finite-time consensus problem for this model — using the $G$ method, a result for reaching a partial or total consensus in finite time is given. Then we consider a special submodel/case of the DeGroot model, and for it we give two examples, and for the latter example (Example 2.11) we give certain comments — a subset/subgroup property of the DeGroot submodel is illustrated in this latter example.

Consider a group (society, team, committee, etc.) of $r$ individuals (experts, agents, etc.) — formally/mathematically we consider a set with $r$ elements, $r \geq 2$; in this article, we consider the set $\langle r \rangle$ ($\langle r \rangle = \{1, 2, ..., r\}$, $r \geq 2$), and we call it the *individual set.* Each individual has an estimation/opinion, an initial one, on a subject (the estimation of a parameter, or of a probability, or...). The initial estimations/opinions of individuals are represented by a (row) vector $p_0 = (p_{0i})_{i \in \langle r \rangle} = (p_{01}, p_{02}, ..., p_{0r})$, $p_{0i}$ is the initial estimation/opinion of individual $i$, $\forall i \in \langle r \rangle$, and $p_0$ is a probability (row) vector, or a nonnegative vector, or a real vector, or a complex vector, or... When the individuals are made aware of each others' opinions, they may modify their own opinion by taking into account the opinion of others, and the modifications, if any, are done at discrete times $1, 2, ...$ — formally, we consider a sequence $(p_n)_{n \geq 0}$ of (row) vectors and a sequence $(P_n)_{n \geq 1}$ of stochastic matrices, $p_n = (p_{ni})_{i \in \langle r \rangle} = (p_{n1}, p_{n2}, ..., p_{nr})$, $\forall n \geq 0$, and $P_n = \left((P_n)_{ij}\right) \in S_r$, $\forall n \geq 1$, where $p_{ni}$ is the estimation/opinion of individual $i$ at time $n$, $\forall n \geq 0$, $\forall i \in \langle r \rangle$, and $(P_n)_{ij}$ is the probability which the individual $i$ assigns to the estimation/opinion $p_{n-1j}$ (warning! not $p_{0j}$) of individual $j$ at time $n$ (not $n-1$), $\forall n \geq 1$, $\forall i, j \in \langle r \rangle$. We call $p_n$ the *vector of estimations/opinions* (*of individuals*) *at time $n$ or the value vector at time $n$*, $\forall n \geq 0$, and $P_n$ the *trust* or *weight matrix at time $n$*, $\forall n \geq 1$. We also call $p_0$ the *initial vector of estimations/opinions* (*of individuals*) *or the initial value vector.* $p_n$ is a probability (row) vector, or a nonnegative vector, or a real vector, or a complex vector, or..., $\forall n \geq 0$. The above considerations lead to

$$p'_n = P_n p'_{n-1}, \ \forall n \geq 1$$

($p'_n$ and $p'_{n-1}$ are the transposes of $p_n$ and $p_{n-1}$, respectively, $\forall n \geq 1$), and, as a result, to

$$p'_n = P_n P_{n-1} ... P_1 p'_0, \ \forall n \geq 1.$$

The above model [7]... is determined by the set $\langle r \rangle$, vector $p_0$, and stochastic matrices $P_n$, $n \geq 1$. Due this fact, we call this model the *DeGroot model on* (*the triple*) $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, and will use the generic name "(the) DeGroot model". We say that DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$ is *homogeneous* if $P_1 = P_2 = P_3 = ...$ and *nonhomogeneous* if $\exists u, v \geq 1$, $u \neq v$, such that $P_u \neq P_v$, and will use the generic names "(the) homogeneous DeGroot model" and "(the) nonhomogeneous DeGroot model", respectively.

**Definition 2.1.** ([7]; see, *e.g.*, also [5].) *We say that the individual set reaches a consensus or that a consensus is reached* (*for the individual set*) *if all $r$ components of $p_n$ converge to the same limit as $n \longrightarrow \infty$, and we call this limit the consensus.*

Reaching a consensus for the DeGroot model — this is the *consensus problem* for/of the DeGroot model. When a consensus is reached, it is reached in a finite or infinite time — obviously, the first case is the best.

The DeGroot model and its consensus problem and the consensus problem in computer science have some things in common with the finite Markov chains — for the first three, see, *e.g.*, [1, Chapter 5, pp. 91−124], [2]-[3], [5]-[7], [9], [12, Chapter 8], [15], [22], [26]-[28], and [30], and for the last, see, *e.g.*, [10]-[11], [13], [16], and [24]-[26] (all or some of these references could be available; arXiv and Wikipedia could also be useful... — expressions for search: DeGroot model, DeGroot learning, DeGroot learning model, consensus (computer science), distributed consensus, distributed averaging, Markov chain, homogeneous Markov chain, inhomogeneous Markov chain, nonhomogeneous Markov chain...).

**Theorem 2.2.** (A simple, important, and known result, see [7], [2]...) *Consider the DeGroot model on* $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$. *If* $\lim_{n\to\infty} P_nP_{n-1}...P_1$ *exists and is a stable matrix, then a consensus is reached.*

*Proof.* (A known proof, but we give it — the best thing is to give it.) Suppose that $\lim_{n\to\infty} P_nP_{n-1}...P_1$ exists and is a stable matrix. In this case, $\exists\pi$, $\pi$ is a probability (row) vector with $r$ components, such that

$$\lim_{n\to\infty} P_nP_{n-1}...P_1 = e'\pi$$

(recall that $e = e(r) = (1, 1, ..., 1)$). Further, we have

$$\lim_{n\to\infty} p'_n = \lim_{n\to\infty} P_nP_{n-1}...P_1p'_0 = e'\pi p'_0 = e'\sum_{i=1}^{r}\pi_i p_{0i} = \begin{pmatrix} \sum_{i=1}^{r}\pi_i p_{0i} \\ \sum_{i=1}^{r}\pi_i p_{0i} \\ \vdots \\ \sum_{i=1}^{r}\pi_i p_{0i} \end{pmatrix}.$$

Therefore, a consensus is reached; the consensus is

$$\sum_{i=1}^{r} \pi_i p_{0i}.$$

∎

**Remark 2.3.** The condition from the above theorem is only a sufficient condition for reaching a consensus. Indeed, if

$$p_0 = (2, 4, 3) \text{ and } P_n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \forall n \geq 1$$

(a homogeneous case), then a consensus is reached at time 1 (a finite-time consensus), the consensus is 3, but

$$\lim_{n\to\infty} P_n P_{n-1} ... P_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is not a stable matrix. Another example, a nonhomogeneous case:

$$p_0 = \left(2, 4, \frac{10}{3}\right), \; P_1 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } P_n = \begin{pmatrix} \multicolumn{2}{c}{Q_n} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & 1 \end{pmatrix}, \forall n \geq 2,$$

where $Q_n$ is a stochastic $2 \times 2$ matrix, $\forall n \geq 2$ — a consensus is reached at time 1, and this is $\frac{10}{3}$. Note that in the above cases/examples the matrices are reducible — therefore, a consensus, even a finite-time consensus, can be reached even when the matrices are reducible.

The above remark is both for Theorem 2.2 and for an incorrect statement in [7, Sec. 4, p. 119] discovered by R.L. Berger [2].

**Remark 2.4.** (a) If $P \in S_{m,n}$ and $Q \in S_{n,p}$, and $P$ is a stable matrix, then $PQ$ is a stable matrix. More generally, if $P \in C_{m,n}$ and $Q \in C_{n,p}$, and $P$ is a stable (complex) matrix, then $PQ$ is a stable matrix.

(b) If $P \in S_{m,n}$ and $Q \in C_{n,p}$, and $Q$ is a stable matrix, then $PQ$ is a stable matrix, and, moreover,

$$PQ = Q.$$

More generally, if $P \in C_{m,n}$ and $Q \in C_{n,p}$, $P$ is a generalized stochastic complex matrix (see Definition 1.3) and $Q$ is a stable (complex) matrix, then $PQ$ is a stable matrix, and, moreover,

$$PQ = Q,$$

but only when $P$ is a stochastic complex matrix.

(c) (simple and important proprieties of the (homogeneous and nonhomogeneous) DeGroot model) Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$.

(c1) If $P_tP_{t-1}...P_m$ is a stable matrix for some $t$ and $m$, $t \geq m \geq 1$, then (see (a) and (b))

$$P_nP_{n-1}...P_1$$

is a stable matrix, $\forall n \geq t$, and, moreover,

$$P_nP_{n-1}...P_1 = P_tP_{t-1}...P_1, \ \forall n \geq t.$$

This fact leads to other ones: 1) the sequence of matrices $(P_n)_{n\geq t+1}$ does not count, it can be replaced with any other infinite sequence of stochastic $r \times r$ matrices; 2) a consensus is reached in a finite time, and it is

$$\sum_{i=1}^{r} \pi_i p_{0i},$$

where $\pi = (\pi_1, \pi_2, ..., \pi_r)$, $\pi = (P_tP_{t-1}...P_1)_{\{i\}}$, $\forall i \in \langle r\rangle$ ($P_tP_{t-1}...P_1$ is a stable matrix, *i.e.*, a matrix with identical rows; $(P_tP_{t-1}...P_1)_{\{i\}}$ is the row $i$ of $P_tP_{t-1}...P_1$, $\forall i \in \langle r\rangle$).

(c2) If $(P_tP_{t-1}...P_1)^K$ is a stable matrix for some $t$ and $K$, $t \geq 1$ and $\emptyset \neq K \subseteq \langle r\rangle$, then (see Remark 1.2(b) and (b))

$$(P_nP_{n-1}...P_1)^K$$

is a stable matrix, $\forall n \geq t$, and, moreover,

$$(P_nP_{n-1}...P_1)^K = (P_tP_{t-1}...P_1)^K, \ \forall n \geq t$$

— $\forall n \geq t$, $P_nP_{n-1}...P_1 = P_tP_{t-1}...P_1$ and $P_nP_{n-1}...P_1$ is a stable matrix when $|K^c| = 0$; $\forall n \geq t$, $(P_nP_{n-1}...P_1)^K = (P_tP_{t-1}...P_1)^K$ and $(P_nP_{n-1}...P_1)^K$ is a stable matrix $\Longrightarrow$ $P_nP_{n-1}...P_1 = P_tP_{t-1}...P_1$ and $P_nP_{n-1}...P_1$ is a stable matrix when $|K^c| = 1$ because the matrices $P_b$, $b \geq 1$, are stochastic; so, $\forall n \geq t$, $P_nP_{n-1}...P_1 = P_tP_{t-1}...P_1$ and $P_nP_{n-1}...P_1$ is a stable matrix when $|K^c| \leq 1$.

Below we give a fundamental result concerning the finite-time partial and total consensuses (for reaching a partial or total consensus in a finite time).

**Theorem 2.5.** *Consider the DeGroot model on* $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$. *If* $P_1 \in G_{\Delta_2,\Delta_1} \subseteq S_r$, $P_2 \in G_{\Delta_3,\Delta_2} \subseteq S_r$ ..., $P_t \in G_{\Delta_{t+1},\Delta_t} \subseteq S_r$, *and* $\Delta_{t+1} = (\langle r\rangle)$ *for some* $t \geq 1$ *and*

$$\Delta_1 = (\{j\}, K^c)_{j\in K} \ (\emptyset \neq K \subseteq \langle r\rangle),$$

*then*

$$(P_tP_{t-1}...P_1)^K$$

(*equivalently,* $P_tP_{t-1}...P_2P_1^K$) *is a stable matrix,*

$$(P_tP_{t-1}...P_1)^K_{\{i\}} = \left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}} = P_t^{-+}P_{t-1}^{-+}...P_2^{-+}\left(P_1^K\right)^{-+}, \forall i \in \langle r\rangle,$$

*where*

$$\left(P_1^K\right)^{-+} = \left(P_1^K\right)^{-+\left(\Delta_2,(\{j\})_{j\in K}\right)}$$

($(P_tP_{t-1}...P_1)^K_{\{i\}}$ *is the row* $i$ *of matrix* $(P_tP_{t-1}...P_1)^K$; $\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}}$ *is a submatrix of* $P_t^{-+}P_{t-1}^{-+}...P_1^{-+}$ (*see the definition of operator* $(\cdot)^{(\cdot)}$ *again*); $P_b^{-+} = P_b^{-+(\Delta_{b+1},\Delta_b)}$, $\forall b \in \langle t\rangle$; *we used* $\{\{j\}\}$ *because the labels of columns of* $P_t^{-+}P_{t-1}^{-+}...P_1^{-+}$ *are, here,* $\{j\}$, $j \in \langle r\rangle$, *when* $|K^c| \le 1$ *and* $\{j\}$, $j \in K$, *and* $K^c$ *when* $|K^c| > 1$ (*see the definition of operator* $(\cdot)^{-+}$ *again*)), *and*

$$(P_tP_{t-1}...P_1)^K = e'\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}} =$$

$$= e'P_t^{-+}P_{t-1}^{-+}...P_2^{-+}\left(P_1^K\right)^{-+},$$

*where* $e = e(r)$. *Further, since*

$$p_n' = P_nP_{n-1}...P_1p_0'$$

*and* (*see Remark* 2.4(c2))

$$(P_nP_{n-1}...P_1)^K = (P_tP_{t-1}...P_1)^K, \ \forall n \ge t, \ \ \textit{if} \ \ |K^c| > 1$$

*and*

$$P_nP_{n-1}...P_1 = P_tP_{t-1}...P_1 \ \ \textit{and...}, \ \forall n \ge t, \ \ \textit{if} \ \ |K^c| \le 1,$$

*we have*

$$p_{nl} = \begin{cases} \sum\limits_{j\in\langle r\rangle}\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)_{\langle r\rangle\{j\}}p_{0j} \\ \qquad\qquad \textit{if} \ \ |K^c| \le 1, \\ \sum\limits_{j\in K}\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)_{\langle r\rangle\{j\}}p_{0j} + \sum\limits_{u\in K^c}(P_nP_{n-1}...P_1)_{lu}\,p_{0u} \\ \qquad\qquad \textit{if} \ \ |K^c| > 1, \end{cases}$$

$\forall n \ge t$, $\forall l \in \langle r\rangle$ (*when* $|K^c| > 1$, *we can replace* $\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)_{\langle r\rangle\{j\}}$ *with* $\left(P_t^{-+}P_{t-1}^{-+}...P_2^{-+}\left(P_1^K\right)^{-+}\right)_{\langle r\rangle\{j\}}$; $P_t^{-+}P_{t-1}^{-+}...P_1^{-+}$ *has just one row, and the label of this row is* $\langle r\rangle$) — *when* $\emptyset \ne K \subseteq \langle r\rangle$ ($K = \langle r\rangle$ *or* $K \ne \langle r\rangle$), *the linear* *combination of* $p_{0j}$, $j \in K$, *namely,* $\sum_{j\in K} \left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)_{\langle r\rangle\{j\}} p_{0j}$, *does not depend on* $n$ *and* $l$ *when* $n \geq t$, *and, therefore, a partial or total consensus is reached at time* $t$ (*the linear combination of* $p_{0j}$, $j \in K$, *at time/step* $t$ *is kept at any time* $n > t$ — *the utilization of phrases "partial consensus" and "total consensus" are thus justified*); *when* $|K^c| \leq 1$, *a* (*total*) *consensus is reached at time* $t$ *and this is*

$$\sum_{j\in\langle r\rangle} \left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)_{\langle r\rangle\{j\}} p_{0j};$$

*when* $|K^c| > 1$,

$$\sum_{j\in K} \left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)_{\langle r\rangle\{j\}} p_{0j}$$

*is a partial consensus reached at time* $t$ *for the initial opinions,* $p_{0j}$, $j \in K$, *of individuals from* $K$ *and the linear combination of* $p_{0u}$, $u \in K^c$, *namely,* $\sum_{u\in K^c} (P_nP_{n-1}...P_1)_{lu}\, p_{0u}$, *depends or not on* $n$ *and* $l$. (See also Remark 1.11.)

*Proof.* Theorems 1.10 and 2.2 and Remarks 1.2(b) and 2.4, (b) or (c). (Using Remark 1.2(b), we have

$$(P_nP_{n-1}...P_1)^K = P_nP_{n-1}...P_{t+1}\left(P_tP_{t-1}...P_1\right)^K = \left(P_tP_{t-1}...P_1\right)^K, \ \forall n \geq t$$

— the latter equation follows from the fact that $(P_tP_{t-1}...P_1)^K$ is a stable (stochastic) matrix ($P_nP_{n-1}...P_{t+1}$ vanishes when $n = t$).) ∎

**Remark 2.6.** (a) $P \in G_{(\{i\})_{i\in\langle n\rangle},(\{i\})_{i\in\langle n\rangle}}$, $\forall P \in S_n$, where $n \geq 1$ (by Remark 1.7(a)).

(b) A case, an interesting one, of Theorem 2.5 is when $\Delta_1 = \Delta_2 = ... = \Delta_u = (\{i\})_{i\in\langle r\rangle}$ and $\Delta_{u+1} \neq (\{i\})_{i\in\langle r\rangle}$ for some $u$, $2 \leq u \leq t$ (due to (a), "$P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, ..., $P_{u-1} \in G_{\Delta_u,\Delta_{u-1}}$" holds without problems). In this case, a (total) consensus is reached.

(c) On Theorem 2.5, our interest is to have a set $K$ as large as possible and a natural number $t \geq 1$ as small as possible — interesting optimization problems.

Below we give two examples to illustrate Theorem 2.5; Remark 2.6(b) is illustrated in the next example.

**Example 2.7.** Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where $r=4$, $p_0=(2,4,1,1)$, and the first four matrices are

$$P_1=\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{2}{4} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{2}{4} & \frac{1}{4} \\ \frac{3}{16} & \frac{1}{16} & \frac{2}{4} & \frac{1}{4} \end{pmatrix},\ P_2=\begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix},$$

$$P_3=\begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{4} & 0 \end{pmatrix},\ \text{and } P_4=\begin{pmatrix} \frac{1}{8} & \frac{2}{8} & \frac{3}{8} & \frac{2}{8} \\ \frac{2}{8} & \frac{3}{8} & \frac{2}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We have $P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, and $P_3 \in G_{\Delta_4,\Delta_3}$, where $\Delta_1 = \Delta_2 = (\{j\})_{j\in\langle 4\rangle}$ (see now Remark 2.6(b)), $\Delta_3 = (\langle 2\rangle, \{3,4\})$, and $\Delta_4 = (\langle 4\rangle)$ ($P_1 \in G_{\Delta_4,\Delta_3}$, but this thing does not count here). By Theorem 2.5 (or by Theorem 1.10), $P_3P_2P_1$ is a stable matrix ($P_3P_2$ is also a stable matrix). Further, we have

$$P_3^{-+}P_2^{-+}P_1^{-+} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{2}{4} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{2}{4} & \frac{1}{4} \\ \frac{3}{16} & \frac{1}{16} & \frac{2}{4} & \frac{1}{4} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{55}{256} & \frac{9}{256} & \frac{88}{256} & \frac{104}{256} \end{pmatrix},$$

so, the consensus is

$$P_3^{-+}P_2^{-+}P_1^{-+}p_0' = \frac{110+36+88+104}{256} = \frac{338}{256}$$

(it is reached at time 3). The sequence of matrices $(P_n)_{n\geq 4}$ does not count — this can be replaced with any other infinite sequence of stochastic $4\times 4$ matrices (see Remark 2.4(c1)).

**Example 2.8.** Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where $r=4$, $p_0=(10,20,2,4)$, and the first two matrices are

$$P_1=\begin{pmatrix} \frac{2}{10} & \frac{1}{10} & \frac{7}{10} & 0 \\ \frac{2}{10} & \frac{1}{10} & \frac{1}{10} & \frac{6}{10} \\ 0 & \frac{3}{10} & 0 & \frac{7}{10} \\ 0 & \frac{3}{10} & \frac{2}{10} & \frac{5}{10} \end{pmatrix} \text{ and } P_2=\begin{pmatrix} \frac{4}{10} & 0 & 0 & \frac{6}{10} \\ \frac{1}{10} & \frac{3}{10} & \frac{4}{10} & \frac{2}{10} \\ \frac{1}{10} & \frac{3}{10} & \frac{1}{10} & \frac{5}{10} \\ \frac{2}{10} & \frac{2}{10} & \frac{1}{10} & \frac{5}{10} \end{pmatrix}.$$

We have $P_1 \in G_{\Delta_2,\Delta_1}$ and $P_2 \in G_{\Delta_3,\Delta_2}$, where $\Delta_1 = (\{1\},\{2\},\{3,4\})$, $\Delta_2 = (\{1,2\},\{3,4\})$, and $\Delta_3 = (\langle 4\rangle)$. So, by Theorem 2.5 (or by Theorem 1.10),

$$(P_2P_1)^{\{1,2\}}$$

is a stable matrix. Further, we have

$$P_2^{-+}P_1^{-+} = \begin{pmatrix} \frac{4}{10} & \frac{6}{10} \end{pmatrix} \begin{pmatrix} \frac{2}{10} & \frac{1}{10} & \frac{7}{10} \\ 0 & \frac{3}{10} & \frac{7}{10} \end{pmatrix} = \begin{pmatrix} \frac{8}{100} & \frac{22}{100} & \frac{70}{100} \end{pmatrix}.$$

So (see Theorem 2.5),

$$p_{nl} = \frac{8}{100}\cdot 10 + \frac{22}{100}\cdot 20 + (P_nP_{n-1}...P_1)_{l3}\cdot 2 + (P_nP_{n-1}...P_1)_{l4}\cdot 4 =$$
$$= 5.2 + 2\,(P_nP_{n-1}...P_1)_{l3} + 4\,(P_nP_{n-1}...P_1)_{l4}\,,\ \forall n \geq 2,\ \forall l \in \langle 4\rangle$$

— the linear combination,

$$\frac{8}{100}\cdot 10 + \frac{22}{100}\cdot 20,$$

for the initial opinions 10 and 20 (of individuals 1 and 2) holds at any time $n \geq 2$; the consensus, this is a partial consensus, for the initial opinions 10 and 20 is 5.2, and is reached at time 2.

Below we consider a possible case of the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$ — a submodel of the DeGroot model which works from small subsets (subgroups of individuals) of $\langle r\rangle$ to larger and larger subsets of $\langle r\rangle$ until the whole set $\langle r\rangle$.

Let $\Delta_1, \Delta_2, ..., \Delta_{t+1} \in\mathrm{Par}(\langle r\rangle)$, $r \geq 2$, $t \geq 1$. Assume that $\Delta_1 \preceq \Delta_2 \preceq ... \preceq \Delta_{t+1}$, $\Delta_1 = (\{j\})_{j\in\langle r\rangle}$, and $\Delta_{t+1} = (\langle r\rangle)$. Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where the matrices $P_1, P_2, ..., P_{t-1}$ (not $P_1, P_2, ..., P_t$) when $t \geq 2$ have the property

$$(P_l)_K^L = 0,\ \forall l \in \langle t-1\rangle, \forall K, L \in \Delta_{l+1},\ K \neq L$$

(this assumption implies that $P_l$ is a block diagonal matrix, eventually by permutation of rows and columns, and $\Delta_{l+1}$-stable matrix on $\Delta_{l+1}$, $\forall l \in \langle t-1\rangle$).

**Remark 2.9.** Theorem 2.5 holds, in particular, for the above DeGroot submodel when $P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, ..., $P_t \in G_{\Delta_{t+1},\Delta_t}$.

To illustrate the DeGroot submodel, we give two examples.

**Example 2.10.** Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where

$$r = 6,\ p_0 = \left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10}, \frac{2}{10}\right),$$

and the first three matrices are

$$P_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{2}{3} & \frac{1}{3} & & & & \\ & & \frac{1}{4} & \frac{3}{4} & & \\ & & \frac{2}{4} & \frac{2}{4} & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}, \ P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & \\ 0 & 1 & 0 & 0 & & \\ & & & & 1 & 0 \\ & & & & \frac{1}{10} & \frac{9}{10} \end{pmatrix},$$

and

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

Taking $\Delta_1 = (\{j\})_{j\in\langle 6\rangle}$, $\Delta_2 = (\langle 2\rangle, \{3,4\}, \{5,6\})$, $\Delta_3 = (\langle 4\rangle, \{5,6\})$, and $\Delta_4 = (\langle 6\rangle)$, we have $\Delta_1 \preceq \Delta_2 \preceq \Delta_3 \preceq \Delta_4$ and

$$(P_1)_{\langle 2\rangle}^{\{3,4\}} \equiv (P_1)_{\langle 2\rangle}^{\{5,6\}} = 0, \ (P_1)_{\{3,4\}}^{\langle 2\rangle} \equiv (P_1)_{\{3,4\}}^{\{5,6\}} = 0,$$

$$(P_1)_{\{5,6\}}^{\langle 2\rangle} \equiv (P_1)_{\{5,6\}}^{\{3,4\}} = 0, \ (P_2)_{\langle 4\rangle}^{\{5,6\}} = 0, \text{ and } (P_2)_{\{5,6\}}^{\langle 4\rangle} = 0$$

— the conditions of the DeGroot submodel are satisfied, and thus we have an example for this submodel. Learning begins with the small subsets (subgroups of individuals) $\langle 2\rangle$, $\{3,4\}$, and $\{5,6\}$ of $\langle 6\rangle$, the larger subsets $\langle 4\rangle$ and $\{5,6\}$ of $\langle 6\rangle$ are then used, then the whole set, $\langle 6\rangle$, is used. Theorem 2.5 cannot be applied for $\Delta_1$, $\Delta_2$, $\Delta_3$, and $\Delta_4$ because $P_1 \notin G_{\Delta_2,\Delta_1}$ (moreover, $P_2 \notin G_{\Delta_3,\Delta_2}$).

In the next example we show that our DeGroot submodel on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$ has a subset/subgroup property when $P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, ..., $P_t \in G_{\Delta_{t+1},\Delta_t}$.

**Example 2.11.** (The subset/subgroup property illustrated.) Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where

$$r = 6, \ p_0 = \left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10}, \frac{2}{10}\right),$$

and the first four matrices are

$$P_1 = \begin{pmatrix} \frac{4}{10} & \frac{6}{10} & & & & \\ \frac{4}{10} & \frac{6}{10} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \frac{1}{10} & \frac{9}{10} \\ & & & & \frac{1}{10} & \frac{9}{10} \end{pmatrix}, \ P_2 = \begin{pmatrix} \frac{3}{10} & \frac{4}{10} & \frac{3}{10} & & & \\ \frac{2}{10} & \frac{5}{10} & \frac{3}{10} & & & \\ \frac{1}{10} & \frac{6}{10} & \frac{3}{10} & & & \\ & & & 1 & & \\ & & & & \frac{2}{10} & \frac{8}{10} \\ & & & & \frac{4}{10} & \frac{6}{10} \end{pmatrix},$$

$$P_3 = \begin{pmatrix} \frac{3}{10} & \frac{2}{10} & \frac{5}{10} & & & \\ \frac{2}{10} & \frac{4}{10} & \frac{4}{10} & & & \\ \frac{1}{10} & \frac{6}{10} & \frac{3}{10} & & & \\ & & & \frac{1}{10} & \frac{1}{10} & \frac{8}{10} \\ & & & \frac{1}{10} & \frac{2}{10} & \frac{7}{10} \\ & & & \frac{1}{10} & \frac{1}{10} & \frac{8}{10} \end{pmatrix}, \ P_4 = \begin{pmatrix} \frac{2}{10} & \frac{2}{10} & \frac{2}{10} & \frac{1}{10} & \frac{1}{10} & \frac{2}{10} \\ \frac{1}{10} & \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \frac{2}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{2}{10} & \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{2}{10} \\ \frac{1}{10} & \frac{4}{10} & \frac{1}{10} & \frac{2}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{4}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{2}{10} \end{pmatrix}.$$

It is easy to see that this model satisfies the conditions of DeGroot submodel for $\Delta_1 = (\{j\})_{j\in\langle 6\rangle}$, $\Delta_2 = (\langle 2\rangle, \{3\}, \{4\}, \{5,6\})$, $\Delta_3 = (\langle 3\rangle, \{4\}, \{5,6\})$, $\Delta_4 = (\langle 3\rangle, \{4,5,6\})$, and $\Delta_5 = (\langle 6\rangle)$. We have $P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, $P_3 \in G_{\Delta_4,\Delta_3}$, and $P_4 \in G_{\Delta_5,\Delta_4}$. By Theorem 2.5, $P_4P_3P_2P_1$ is a stable matrix, so, a consensus is reached at time 4... — the continuation, using Theorem 2.5, is left to the reader (see also Example 2.7). Further, we comment on the matrices $P_1$, $P_2$, $P_3$, and $P_4$, and use Theorem 1.16. An interpretation of $P_1$ (at time 1 — see the DeGroot model again): the individuals 1 and 2 negotiate — negotiation, a good idea —, and agree the same probability, $\frac{4}{10}$, for the initial opinion/estimation of individual 1, *i.e.*, for $p_{01}$, and, separately, the same probability, $\frac{6}{10}$, for the initial opinion of individual 2, *i.e.*, for $p_{02}$; the individuals 5 and 6 proceed similarly; the individuals 3 and 4 do not negotiate... An interpretation of $P_2$ (at time 2): the individuals 1, 2, and 3 negotiate, and agree the same probability, $\frac{7}{10}$ ($\frac{7}{10} = \frac{3}{10}+\frac{4}{10} = \frac{2}{10}+\frac{5}{10} = \frac{1}{10}+\frac{6}{10}$), for the opinions of individuals 1 and 2 at time 1 considered together, *i.e.*, for $p_{11}$ and $p_{12}$ considered together, and, separately, the same probability, $\frac{3}{10}$, for the opinion of individual 3 at time 1, *i.e.*, for $p_{13}$; the individuals 5 and 6 renegotiate, and agree the same probability, 1 ($1 = \frac{2}{10} + \frac{8}{10} = \frac{4}{10} + \frac{6}{10}$), for their opinions at time 1 considered together, *i.e.*, for $p_{15}$ and $p_{16}$ considered together; the individual 4 does not negotiate... An interpretation of $P_3$ (at time 3): the individuals 1, 2, and 3 renegotiate, and agree the same probability, 1 ($1 = \frac{3}{10} + \frac{2}{10} + \frac{5}{10} = \frac{2}{10} + \frac{4}{10} + \frac{4}{10} = \frac{1}{10} + \frac{6}{10} + \frac{3}{10}$), for their opinions at time 2 considered together, *i.e.*, for $p_{21}$, $p_{22}$, and $p_{23}$ considered together; the individuals 4, 5, and 6 negotiate, and agree the same probability, $\frac{1}{10}$, for the opinion of individual 4 at time 2, *i.e.*, for $p_{24}$, and, separately, the same probability, $\frac{9}{10}$ ($\frac{9}{10} = \frac{1}{10}+\frac{8}{10} = \frac{2}{10}+\frac{7}{10} = \frac{1}{10}+\frac{8}{10}$), for the opinions of individuals 5 and 6 at time 2 considered together, *i.e.*, for $p_{25}$ and $p_{26}$ considered together. An interpretation of $P_4$ (at time 4): all the individuals negotiate, and agree the same probability, $\frac{6}{10}$ ($\frac{6}{10} = \frac{2}{10} + \frac{2}{10} + \frac{2}{10} = \frac{1}{10} + \frac{3}{10} + \frac{2}{10} = ...$), for the opinions of individuals 1, 2, and 3 at time 3 considered together, *i.e.*, for $p_{31}$, $p_{32}$, and $p_{33}$ considered together, and, separately, the same probability, $\frac{4}{10}$ ($\frac{4}{10} = \frac{1}{10} + \frac{1}{10} + \frac{2}{10} = \frac{1}{10} + \frac{2}{10} + \frac{1}{10} = ...$), for the opinions of individuals 4, 5, and 6 at time 3 considered together, *i.e.*, for $p_{34}$, $p_{35}$, and $p_{36}$ considered together. The negotiations and renegotiations took place from small subsets (of $\langle 6\rangle$)/subgroups to larger and larger subsets/subgroups until to the whole set/group. By Theorem 1.16, $P_2$ can be replaced with any matrix which is similar to it, such as,

$$\begin{pmatrix} \frac{3}{10} & \frac{4}{10} & \frac{3}{10} & & & \\ \frac{3}{10} & \frac{4}{10} & \frac{3}{10} & & & \\ \frac{3}{10} & \frac{4}{10} & \frac{3}{10} & & & \\ & & & 1 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix} := P_2',$$

and

$$\begin{pmatrix} \frac{7}{10} & 0 & \frac{3}{10} & & & \\ 0 & \frac{7}{10} & \frac{3}{10} & & & \\ \frac{7}{10} & 0 & \frac{3}{10} & & & \\ & & & 1 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} := P_2''.$$

The matrices $(P_2)_{\langle 3\rangle}^{\langle 2\rangle}$,

$$(P_2)_{\langle 3\rangle}^{\langle 2\rangle} = \begin{pmatrix} \frac{3}{10} & \frac{4}{10} \\ \frac{2}{10} & \frac{5}{10} \\ \frac{1}{10} & \frac{6}{10} \end{pmatrix},$$

$(P_2')_{\langle 3\rangle}^{\langle 2\rangle}$,

$$(P_2')_{\langle 3\rangle}^{\langle 2\rangle} = \begin{pmatrix} \frac{3}{10} & \frac{4}{10} \\ \frac{3}{10} & \frac{4}{10} \\ \frac{3}{10} & \frac{4}{10} \end{pmatrix},$$

and $(P_2'')_{\langle 3\rangle}^{\langle 2\rangle}$,

$$\left(P_2''\right)_{\langle 3\rangle}^{\langle 2\rangle} = \begin{pmatrix} \frac{7}{10} & 0 \\ 0 & \frac{7}{10} \\ \frac{7}{10} & 0 \end{pmatrix}$$

(each of them has 6 entries,

$$\begin{pmatrix} \star & \star \\ \star & \star \\ \star & \star \end{pmatrix},$$

"$\star$" stands for a positive or zero entry), have the property, a subset/subgroup property: the row sums are equal to $\frac{7}{10}$ — only the row sums count, not the positions of positive entries, but at least one entry must be positive in each row (because $\frac{7}{10} > 0$), only $\frac{7}{10}$ counts, $\frac{7}{10}$ is the probability... (see above), and this probability is for the subset $\langle 2\rangle$ of $\langle 6\rangle$. Among $P_2$, $P_2'$, and $P_2''$, $P_2''$ is the best because it has the smallest number of positive entries — these are the positive probabilities for the subsets involved here: $\frac{7}{10}$ for (the subset) $\langle 2\rangle$ ($\frac{7}{10}$ = the probability assigned by the individual $i$ for the opinions of individuals 1 and 2 at time 1 considered together, $\forall i \in \langle 3\rangle$), $\frac{3}{10}$ for $\{3\}$, 1 for $\{4\}$, and 1 for $\{5, 6\}$. Similarly, $P_3$ and $P_4$ can be replaced with any matrices, $Q_3$ and $Q_4$, which are similar to $P_3$ and $P_4$, respectively. By Theorem 1.16 we have

$$P_4P_3P_2P_1 = P_4P_3P_2'P_1 = P_4P_3P_2''P_1 = P_4Q_3P_2P_1 = ...$$

(we can work with $P_4P_3P_2P_1$, or with $P_4P_3P_2'P_1$, or with...). Note that $P_1$, $P_2$, and $P_3$ are reducible while $P_4$ is irreducible (being positive), but the last matrix can be replaced with any reducible matrix which is similar to it, such as,

$$\begin{pmatrix} \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \end{pmatrix} := U_4$$

— $P_1$, $P_2$, $P_3$, and $U_4$ are reducible, but, interestingly, $U_4P_3P_2P_1$ is irreducible (because it is positive — it is also stable).

For other examples for the DeGroot submodel, see, in the next section, Remark 3.7.

## 3. FIINITE-TIME CONSENSUS FOR DEGROOT MODEL ON DISTRIBUTED SYSTEMS

In this section, we consider the finite-time consensus problem for the DeGroot model on distributed systems (for these systems, see, *e.g.*, [6], see, *e.g.*, also [4]) — this section can also be useful for the study of other systems, such as, the multi-agent systems (for the multi-agent systems, see, *e.g.*, [14], [23], and [29]). For the DeGroot model on distributed systems, using the $G$ method too, we have a result for reaching a partial or total (distributed) consensus in a finite time similar to that for the DeGroot model for reaching a partial or total consensus in a finite time (see Remark 3.2(b)). We show that for any connected graph having $2^m$ vertices, $m \geq 1$, and a spanning subgraph isomorphic to the $m$-cube graph, distributed averaging is performed in $m$ steps — this result can be extended — research work — for any graph with $n_1 n_2 ... n_t$ vertices under certain conditions, where $t, n_1, n_2, ..., n_t \geq 2$, and, in this case, distributed averaging is performed in $t$ steps.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a (nondirected simple finite) connected graph, where $\mathcal{V}$ is the vertex/node set and $\mathcal{E}$ is the edge set. (A *simple graph* is a graph without multiple edges and loops.) Suppose that $\mathcal{V} = \langle r \rangle$, and that $r \geq 2$. The edge set $\mathcal{E}$ is a set of 2-element subsets of $\langle r \rangle$, so,

$$\mathcal{E} \subseteq \{\{i, j\} \mid i, j \in \langle r \rangle, \ i \neq j\} \ \ (\mathcal{E} \neq \emptyset)$$

— *e.g.*, $\mathcal{E} = \{\{1, 2\}, \{1, 3\}, ..., \{1, r\}\}$ ($r \geq 2$). This graph, on the distributed systems or not, can be a mathematical model for a network of computers, processors, or sensors, for a team, when the graphs are involved, of robots, aircraft, or spacecraft... Set

$$\mathcal{N}_i = \{j \mid j \in \langle r \rangle \text{ and } \{i, j\} \in \mathcal{E}\}, \ \forall i \in \langle r \rangle.$$

$\mathcal{N}_i$ is called the *set of neighbors of* $i$, $\forall i \in \langle r \rangle$. $i \notin \mathcal{N}_i$, $\forall i \in \langle r \rangle$. Suppose that each vertex/node $i \in \langle r \rangle$ holds an initial value, say, $q_{0i}$, $q_{0i} \in \mathbb{R}$, and we call the vector

$$q_0 = (q_{0i})_{i \in \langle r \rangle} = (q_{01}, q_{02}, ..., q_{0r})$$

the *initial value vector*. We use homogeneous and nonhomogeneous linear iterations: at each time $n \geq 1$ ($n \in \mathbb{N}^*$), each vertex $i$ updates its value as (see, *e.g.*, [27] and [30] — only the homogeneous linear iterations are considered in these articles)

$$q_{ni} = \sum_{j \in \{i\} \cup \mathcal{N}_i} (W_n)_{ij} \, q_{n-1j},$$

where $(W_n)_{ij} \in \mathbb{R}$, $\forall j \in \{i\} \cup \mathcal{N}_i$, $(W_n)_{ij}$ is the weight on $q_{n-1j}$ at vertex $i$ at time $n$, $\forall j \in \{i\} \cup \mathcal{N}_i$. Set the (row) vector

$$q_n = (q_{ni})_{i \in \langle r \rangle} = (q_{n1}, q_{n2}, ..., q_{nr}), \ \forall n \geq 0$$

— using $q_0$, we obtain $q_1$ (see above), using $q_1$, we obtain $q_2$... We call the vector $q_n$ the *value vector at time/step* $n$, $\forall n \geq 0$ — the value vector at time 0 is also (see above) called the initial value vector. Set the real $r \times r$ matrix

$$W_n = \left( (W_n)_{ij} \right)_{i,j \in \langle r \rangle},$$

$(W_n)_{ij}$ was defined above, $\forall i \in \langle r \rangle$, $\forall j \in \{i\} \cup \mathcal{N}_i$, and

$$(W_n)_{ij} = 0 \text{ if } j \notin \{i\} \cup \mathcal{N}_i, \ \forall i \in \langle r \rangle$$

(we can have $(W_n)_{ij} = 0$ even when $j \in \{i\} \cup \mathcal{N}_i$, where $i \in \langle r \rangle$ and $n \geq 1$), $\forall n \geq 1$. We call the matrix $W_n$ the *weight matrix at time/step* $n$, $\forall n \geq 1$. $W_n$, $n$ fixed, is symmetric or not (each (nondirected/undirected) edge can be considered as two directed edges). Obviously, we have

$$q_n' = W_n q_{n-1}' = W_n W_{n-1} ... W_1 q_0', \ \forall n \geq 1$$

($q_n'$ is the transpose of $q_n$...).

The above model is a DeGroot-type model — compare this model with the DeGroot model from Section 2. It is determined by the graph $\mathcal{G}$, real vector $q_0$, and real matrices $W_n$, $n \geq 1$. Due these facts, we call it the *DeGroot model on* (*the triple*) $\left( \mathcal{G}, q_0, (W_n)_{n\geq 1} \right)$, and will use the generic name "(the) DeGroot model on distributed systems". We say that the DeGroot model on $\left( \mathcal{G}, q_0, (W_n)_{n\geq 1} \right)$ is *homogeneous* if $W_1 = W_2 = W_3 = ...$ and *nonhomogeneous* if $\exists u, v \geq 1$, $u \neq v$, such that $W_u \neq W_v$, and will use the generic names "(the) homogeneous DeGroot model on distributed systems" and "(the) nonhomogeneous DeGroot model on distributed systems", respectively.

Note that the DeGroot model on $\left( \langle r \rangle, p_0, (P_n)_{n\geq 1} \right)$ (see Section 2) is trivially identical with the DeGroot model on $\left( \mathcal{K}_r, p_0, (P_n)_{n\geq 1} \right)$, $\forall r \geq 2$, $\forall p_0$, $p_0$ is a real vector here, $\forall (P_n)_{n\geq 1}$, $(P_n)_{n\geq 1}$ is a sequence of stochastic matrices here, where $\mathcal{K}_r$ is the complete graph with $r$ vertices, $\forall r \geq 2$ — moreover, it is identical with the DeGroot model on $\left( \mathcal{G}_1, p_0, (P_n)_{n\geq 1} \right)$, $\forall r \geq 2$, $\forall p_0$, $p_0$ is a real vector here, $\forall (P_n)_{n\geq 1}$, $(P_n)_{n\geq 1}$ is a sequence of stochastic matrices here, where $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$, $\mathcal{V}_1 = \langle r \rangle$ and

$$\mathcal{E}_1 = \left\{ \{i,j\} \ \middle| \ i,j \in \langle r \rangle, \ i \neq j, \text{ and } \exists n \geq 1 \text{ such that } (P_n)_{ij} > 0 \text{ or } (P_n)_{ji} > 0 \right\}.$$

Definition 2.1 (from Section 2) and the first part of the next definition are similar.

**Definition 3.1.** (See also [27] and [30].) *Consider the DeGroot model on* $\left( \mathcal{G}, q_0, (W_n)_{n\geq 1} \right)$. *We say that the graph* $\mathcal{G}$ *reaches a* (*distributed*) *consensus* *if* $\lim_{n\to\infty} W_n W_{n-1}...W_1$ *exists and is a stable matrix. Set*

$$W_\infty = \lim_{n\to\infty} W_n W_{n-1}...W_1$$

*when* $\lim_{n\to\infty} W_n W_{n-1}...W_1$ *exists.*($W_\infty$ *is a real* $r \times r$ *matrix.*) *In particular, if*

$$W_\infty = e' \left( \frac{1}{r}, \frac{1}{r}, ..., \frac{1}{r} \right),$$

*we say that the graph* $\mathcal{G}$ *performs distributed averaging* ($e = e(r) = (1, 1, ..., 1)$; $e'$ *is its transpose*).

Reaching a distributed consensus for the DeGroot model on distributed systems — this is the (*distributed*) *consensus problem* for/of the DeGroot model on distributed systems, and is similar to the consensus problem for the DeGroot model. When a distributed consensus is reached, it is reached in a finite or infinite time — obviously, the first case is the best. When distributed averaging is performed, it is performed in a finite or infinite time — the first case is also the best.

When the graph $\mathcal{G}$ performs distributed averaging, each vertex holds the average of initial values, $\frac{1}{r} \sum_{i=1}^{r} q_{0i}$ — an interpretation (see, *e.g.*, [30, p. 65]): if $q_{0i}$ is the amount of a resource at the vertex $i$, $\forall i \in \langle r \rangle$, then the average is the fair or uniform allocation of the resource across the graph; an application (see, *e.g.*, [22, p. 228]): clock synchronization (by averaging) in distributed systems.

For more information concerning the above things, see, *e.g.*, [15], [27], and [30].

Recall that the DeGroot model and its consensus problem and the consensus problem in computer science have some things in common with the finite Markov chains — some references are given in the first paragraph before Theorem 2.2 (in Section 2).

**Remark 3.2.** (a) Concerning the (total) distributed consensuses, obviously, we have a theorem similar to Theorem 2.2 and a remark similar to Remark 2.3.

(b) Concerning the finite-time partial and total distributed consensuses (for reaching a partial or total distributed consensus in a finite time), only when the matrices at times $t+1$, $t+2$, ... are (see Definition 1.3) stochastic real (the matrices at times 1, 2, ..., $t$ being (stochastic or nonstochastic) real), we have a theorem similar to Theorem 2.5 and remarks similar to Remarks 2.4(c) and 2.6(b)-(c).

(c) Concerning distributed averaging, if we construct/find $t$ matrices, $W_1$, $W_2$, ..., $W_t$ ($t \geq 1$), such that

$$W_t W_{t-1} ... W_1 = e' \left( \frac{1}{r}, \frac{1}{r}, ..., \frac{1}{r} \right),$$

it makes no sense to construct other matrices — however, if the reader wants to construct a sequence of matrices, $(W_n)_{n \geq 1}$, such that $W_\infty = W_t W_{t-1} ... W_1$, she/he can take, *e.g.*, $W_{t+1} = W_{t+2} = ... = I_r$, where $I_r$ is the identity matrix of dimension $r$.

Below we give a result for finite-time (total) distributed averaging — an application of the $G$ method (more precisely, of Theorem 1.10)...

**Theorem 3.3.** *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be a* (*nondirected simple finite*) *connected graph. Suppose that* $|\mathcal{V}| = 2^m$, $m \geq 1$. *Suppose that* $\mathcal{G}$ *has a spanning subgraph isomorphic to the* $m$*-cube graph,* $\mathcal{Q}_m$. *Then there exist the weight matrices* $W_1$, $W_2$, ..., $W_m$ *such that distributed averaging is performed for* $\mathcal{G}$ *in* $m$ *steps* (*for any initial value vector* $q_0$). (See Remark 3.2(c) again.)

*Proof.* It is sufficient to consider the case when $\mathcal{G} = \mathcal{Q}_m$ — Theorem 3.3 gives the best possible number of steps, $m$, for distributed averaging for this graph because the diameter of $\mathcal{Q}_m$ is equal to $m$. Further, we consider this case, and using the $G$ method, we show that distributed averaging is performed in $m$ steps.

Consider that $\mathcal{V} = \{0,1\}^m$ and

$$\mathcal{E} = \{\{x, y\} \mid x, y \in \{0,1\}^m \text{ and } H(x,y) = 1\},$$

where $H$ is the Hamming distance (the edge set $\mathcal{E}$ is a set of 2-element subsets of $\{0,1\}^m$; $x = (x_1, x_2, ..., x_m)$ ...). Consider the stochastic matrices — stochastic matrices here! — $W_1, W_2, ..., W_m$ (see Remark 3.2(c) again),

$$(W_l)_{(x_1,x_2,...,x_m)(x_1,x_2,...,x_m)} =$$

$$= (W_l)_{(x_1,x_2,...,x_m)(x_1,x_2,...,x_{m-l},(x_{m-l+1}+1) \bmod 2, x_{m-l+2},...,x_m)} = \frac{1}{2},$$

$\forall l \in \langle m \rangle$, $\forall (x_1, x_2, ..., x_m) \in \{0,1\}^m$ ($(W_l)_{(x_1,x_2,...,x_m)(x_1,x_2,...,x_m)}$ is the entry $((x_1, x_2, ..., x_m), (x_1, x_2, ..., x_m))$ of $W_l$...; $x_{m-l+2}, x_{m-l+3}, , ..., x_m$ vanish when $l = 1$; $x_1, x_2, ..., x_{m-l}$ vanish when $l = m$; obviously, $W_1, W_2, ..., W_m$ are symmetric matrices). Let

$$U_{(x_1,x_2,...,x_v)} =$$

$$= \{(y_1, y_2, ..., y_m) \mid (y_1, y_2, ..., y_m) \in \{0,1\}^m \text{ and } y_1 = x_1, y_2 = x_2, ..., y_v = x_v\},$$

$\forall v \in \langle m \rangle$ . Consider the partitions

$$\Delta_t = \left(U_{(x_1,x_2,\ldots,x_{m-t+1})}\right)_{(x_1,x_2,\ldots,x_{m-t+1})\in\{0,1\}^{m-t+1}},\ \forall t \in \langle m \rangle ,$$

and

$$\Delta_{m+1} = (\{0,1\}^m)$$

of $\{0,1\}^m$ . Obviously,

$$\Delta_1 = \left(U_{(x_1,x_2,\ldots,x_m)}\right)_{(x_1,x_2,\ldots,x_m)\in\{0,1\}^m} = (\{x\})_{x\in\{0,1\}^m} .$$

We have $W_l \in G_{\Delta_{l+1},\Delta_l}, \forall l \in \langle m \rangle$ — we prove this statement. Let $l \in \langle m \rangle$ and $K \in \Delta_{l+1}$. It follows that

$$K = \begin{cases} \{0,1\}^m & \text{if } l = m, \\ U_{(x_1,x_2,\ldots,x_{m-l})} \text{ for some } (x_1,x_2,\ldots,x_{m-l}) \in \{0,1\}^{m-l} & \text{if } l \in \langle m-1 \rangle . \end{cases}$$

*Case* 1. $l = m$. In this case, $\Delta_{m+1} = (\{0,1\}^m)$ and $\Delta_m = \left(U_{(0)}, U_{(1)}\right)$ . Let $(x_1,x_2,\ldots,x_m) \in \{0,1\}^m$ . Since $\{0,1\}^m = U_{(0)} \cup U_{(1)}$, it follows that

$$(x_1,x_2,\ldots,x_m) \in U_{(0)} \text{ or } (x_1,x_2,\ldots,x_m) \in U_{(1)}.$$

If $(x_1,x_2,\ldots,x_m) \in U_{(0)}$, then

$$((x_1+1) \bmod 2, x_2,\ldots,x_m) \in U_{(1)};$$

if $(x_1,x_2,\ldots,x_m) \in U_{(1)}$, then

$$((x_1+1) \bmod 2, x_2,\ldots,x_m) \in U_{(0)}.$$

Further, by the definition of $W_m$, we have $W_m \in G_{\Delta_{m+1},\Delta_m}$.

*Case* 2. $l \in \langle m-1 \rangle$ . In this case,

$$K = U_{(x_1,x_2,\ldots,x_{m-l})} \text{ for some } (x_1,x_2,\ldots,x_{m-l}) \in \{0,1\}^{m-l} .$$

Further, we have $K = U_{(x_1,x_2,\ldots,x_{m-l},0)} \cup U_{(x_1,x_2,\ldots,x_{m-l},1)}$, where, obviously, $U_{(x_1,x_2,\ldots,x_{m-l},0)}, U_{(x_1,x_2,\ldots,x_{m-l},1)} \in \Delta_l$. Let $(x_1,x_2,\ldots,x_m) \in K$. It follows that

$$(x_1,x_2,\ldots,x_m) \in U_{(x_1,x_2,\ldots,x_{m-l},0)} \text{ or } (x_1,x_2,\ldots,x_m) \in U_{(x_1,x_2,\ldots,x_{m-l},1)}.$$

If $(x_1,x_2,\ldots,x_m) \in U_{(x_1,x_2,\ldots,x_{m-l},0)}$, then

$$x_{m-l+1} = 0 \text{ and } (x_1,x_2,\ldots,x_{m-l},1,x_{m-l+2},\ldots,x_m) \in U_{(x_1,x_2,\ldots,x_{m-l},1)};$$

if $(x_1, x_2, ..., x_m) \in U_{(x_1,x_2,...,x_{m-l},1)}$, then

$$x_{m-l+1} = 1 \text{ and } (x_1, x_2, ..., x_{m-l}, 0, x_{m-l+2}, ..., x_m) \in U_{(x_1,x_2,...,x_{m-l},0)}.$$

Further, by the definition of $W_l$, we have $W_l \in G_{\Delta_{l+1},\Delta_l}$. The statement was proved. Further, since $W_l \in G_{\Delta_{l+1},\Delta_l}$, $\forall l \in \langle m \rangle$, by Theorem 1.10, $W_m W_{m-1} ... W_1$ is a stable matrix and

$$W_m W_{m-1} ... W_1 = e' W_m^{-+} W_{m-1}^{-+} ... W_1^{-+}.$$

Since

$$W_m^{-+} W_{m-1}^{-+} ... W_1^{-+} =$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} ... \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & \\ & & & & \ddots & & \\ & & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2^m} & \frac{1}{2^m} & ... & \frac{1}{2^m} \end{pmatrix},$$

we have

$$q_m = \left(W_m W_{m-1} ... W_1 q_0'\right)' = \left(\frac{q_{0(0,0,...,0)}}{2^m}, \frac{q_{0(0,0,...,0,1)}}{2^m}, ..., \frac{q_{0(1,1,...,1)}}{2^m}\right)$$

($\mathcal{V} = \{0,1\}^m$, so, $|\mathcal{V}| = 2^m$; $q_{0(0,0,...,0)}$ is the component $(0, 0, ..., 0)$ of (initial value vector) $q_0$...). So, distributed averaging is performed in $m$ steps. ∎

**Example 3.4.** To illustrate Theorem 3.3, we consider the 3-cube graph, $\mathcal{Q}_3$. In this case, we have $\mathcal{V} = \{0,1\}^3$ and

$$\mathcal{E} = \left\{ \{x, y\} \,\middle|\, x, y \in \{0,1\}^3 \text{ and } H(x,y) = 1 \right\} =$$

$$= \{\{(0,0,0),(0,0,1)\}, \{(0,0,0),(0,1,0)\}, ..., \{(1,1,0),(1,1,1)\}\}.$$

Further, we have

$$W_1 = \begin{array}{c} (0,0,0) \\ (0,0,1) \\ (0,1,0) \\ (0,1,1) \\ (1,0,0) \\ (1,0,1) \\ (1,1,0) \\ (1,1,1) \end{array} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & & \\ & & & & \frac{1}{2} & \frac{1}{2} & & \\ & & & & \frac{1}{2} & \frac{1}{2} & & \\ & & & & & & \frac{1}{2} & \frac{1}{2} \\ & & & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(the columns are labeled similarly, *i.e.*, using the lexicographic order from left to right),

$$W_2 = \begin{array}{c} (0,0,0) \\ (0,0,1) \\ (0,1,0) \\ (0,1,1) \\ (1,0,0) \\ (1,0,1) \\ (1,1,0) \\ (1,1,1) \end{array} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & & & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & & & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & & & & \\ & & & & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ & & & & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ & & & & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

$$W_3 = \begin{array}{c} (0,0,0) \\ (0,0,1) \\ (0,1,0) \\ (0,1,1) \\ (1,0,0) \\ (1,0,1) \\ (1,1,0) \\ (1,1,1) \end{array} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Further, we have

$$U_{(x_1,x_2,x_3)} = \{(x_1,x_2,x_3)\}\,,\ \forall\,(x_1,x_2,x_3) \in \{0,1\}^3\,,$$

$$U_{(0,0)} = \{(0,0,0)\,,(0,0,1)\}\,,U_{(0,1)} = \{(0,1,0)\,,(0,1,1)\}\,,$$

$$U_{(1,0)} = \{(1,0,0)\,,(1,0,1)\}\,,U_{(1,1)} = \{(1,1,0)\,,(1,1,1)\}\,,$$

$$U_{(0)} = U_{(0,0)} \cup U_{(0,1)} = \{(0,0,0)\,,(0,0,1)\,,(0,1,0)\,,(0,1,1)\}\,,$$

$$U_{(1)} = U_{(1,0)} \cup U_{(1,1)} = \{(1,0,0)\,,(1,0,1)\,,(1,1,0)\,,(1,1,1)\}\,.$$

Further, we have

$$\Delta_1 = (\{(x_1,x_2,x_3)\})_{(x_1,x_2,x_3)\in\{0,1\}^3} = (\{(0,0,0)\}\,,\{(0,0,1)\}\,,...,\{(1,1,1)\})\,,$$

$$\Delta_2 = \left(U_{(0,0)},U_{(0,1)},U_{(1,0)},U_{(1,1)}\right),$$

$$\Delta_3 = \left(U_{(0)},U_{(1)}\right),$$

$$\Delta_4 = \left(\{0,1\}^3\right)$$

($\Delta_4$ is the improper (degenerate) partition of $\{0,1\}^3$ — $\Delta_4$ has only one set). Further, we have $W_1 \in G_{\Delta_2,\Delta_1}$, $W_2 \in G_{\Delta_3,\Delta_2}$, $W_3 \in G_{\Delta_4,\Delta_3}$; further, we have

$$W_3^{-+}W_2^{-+}W_1^{-+} =$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & & \\ & & & & \frac{1}{2} & \frac{1}{2} & & \\ & & & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2^3} & \frac{1}{2^3} & \cdots & \frac{1}{2^3} \end{pmatrix}.$$

By Theorem 1.10 ($W_3W_2W_1 = e'W_3^{-+}W_2^{-+}W_1^{-+}$) or by direct computation,

$$W_3W_2W_1 = \begin{pmatrix} \frac{1}{2^3} & \frac{1}{2^3} & \cdots & \frac{1}{2^3} \\ \frac{1}{2^3} & \frac{1}{2^3} & \cdots & \frac{1}{2^3} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{2^3} & \frac{1}{2^3} & \cdots & \frac{1}{2^3} \end{pmatrix}.$$

Finally, we have ($\mathcal{V} = \{0,1\}^3$, so, $|\mathcal{V}| = 2^3$; $q_{0(0,0,0)}$ is the component $(0,0,0)$ of $q_0$...)

$$q_3 = \left(W_3W_2W_1q_0'\right)' = \left(\frac{q_{0(0,0,0)}}{2^3}, \frac{q_{0(0,0,1)}}{2^3}, ..., \frac{q_{0(1,1,1)}}{2^3}\right)$$

— distributed averaging is performed at time/step 3 (for any initial value vector $q_0$). To understand Theorem 3.3 and this example better, the reader, if she/he wants, can compute the value vectors $q_1$ and $q_2$. Note that the matrices $W_1, W_2$, and $W_3$ are symmetric, but by Theorem 1.16, $W_2$ and $W_3$ can be replaced with nonsymmetric ones.

**Remark 3.5.** (Research work — find the graphs.) Theorem 3.3 can be extended for any connected graph with $n_1n_2...n_t$ vertices, where $n_1$, $n_2$, ..., $n_t \geq 2$ (preferably prime numbers), and having a spanning subgraph for which Theorem 1.10 can be applied for $t$ stochastic matrices... — we will perform distributed averaging in $t$ steps.

For the above remark, we consider the next example.

**Example 3.6.** (Suggested by Example 2.11 from [17], which refers to the uniform generation of permutations of order $n$.) Consider the graph $\mathcal{G}_3 = (\mathcal{V}_3, \mathcal{E}_3)$, where $\mathcal{V}_3 = \mathbb{S}_3$, $\mathbb{S}_3$ = the set of permutations of order 3, and $\mathcal{E}_3$ = the union of edge sets of nondirected graphs of matrices $W_1$ and $W_2$, where

$$W_1 = \begin{array}{c} (123) \\ (132) \\ (213) \\ (231) \\ (321) \\ (312) \end{array} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} & & \\ & & & & \frac{1}{2} & \frac{1}{2} \\ & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$W_2 = \begin{array}{c} (123) \\ (132) \\ (213) \\ (231) \\ (321) \\ (312) \end{array} \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

The matrices $W_1$ and $W_2$ were obtained by the swapping method — for this method, see, *e.g.*, [8, pp. 645–646]. By Theorem 1.10 ($W_1 \in G_{\Delta_2,\Delta_1}$, $W_2 \in G_{\Delta_3,\Delta_2}$, where...) or by direct computation,

$$W_2W_1 = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \cdots & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \cdots & \frac{1}{6} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{6} & \frac{1}{6} & \cdots & \frac{1}{6} \end{pmatrix},$$

so, distributed averaging is performed, in this case, at time 2 ($3! = 6 = 2 \cdot 3$). Note that the graph $\mathcal{G}_3$ is a 3-regular graph and is not isomorphic to the triangular prism graph, which is also a 3-regular graph — $\mathcal{G}_3$ can be replaced with any graph with 3! vertices which has a spanning subgraph isomorphic to $\mathcal{G}_3$. For the triangular prism graph, using the $G$ method, distributed averaging can also be performed at time 2 — an exercise for the reader.

The above example can be generalized for $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$, $\mathcal{V}_n = \mathbb{S}_n$, $\mathbb{S}_n =$ the set of permutations of order $n$... Due to the swapping method, this graph must be an $\frac{n(n-1)}{2}$ - regular graph — a necessary but not sufficient condition (see the above example).

**Remark 3.7.** (a) For the DeGroot model on distributed systems, labelling the vertices of (involved) graphs is an important problem — a good/suitable labelling can lead to a good/fast algorithm (see Theorem 3.3 and its proof and Examples 3.4 and 3.6).

(b) Theorem 3.3 and its proof and Example 3.6 (see also Remark 3.5) lead to possible cases for the DeGroot submodel (from Section 2) on graphs/networks, in each case, a consensus being reached in a finite time — for the DeGroot model on graphs, see, *e.g.*, [9] and [12].

For the distributed consensuses (in particular, for distributed averaging), we could combine the $G$ method with other methods, obtaining certain hybrid methods: we could use the $G$ method for subgraphs, we could use the $G$ method together with the flooding (for this method, see, *e.g.* [30, p. 65], see, *e.g.*, also [22, p. 228]), we could use a leader vertex or more... This idea can be developed — another research work —, but we give only an example, the next example.

**Example 3.8.** Consider the graph from Fig. 1 in [30] (this graph is also considered in [27, Fig. 1]) — the weights from there are not taken into account. This graph, say, $\mathcal{G}$, can be obtained from the wheel graph $\mathcal{W}_5$ and complete graph $\mathcal{K}_5$ by edge merging. Consider that the vertices of $\mathcal{K}_5$ are 1, 2, 3, 4, 5 and the vertices of $\mathcal{W}_5$ are 1, 2, 6, 7, 8, and that 8 is the universal vertex of $\mathcal{W}_5$, 6 is adjacent to 1 (6 is also adjacent to 8), and 7 is adjacent to 2 (7 is also adjacent to 6 and 8)— $\{1,2\}$ is the common edge of $\mathcal{W}_5$ and $\mathcal{K}_5$. Consider that the initial value vector is $q_0$. First, we consider the subgraph $\mathcal{W}_5$ of $\mathcal{G}$. For this subgraph, we consider that the initial value vector is $q_0^{(1)} = q_0 \mid_{\{1,2,6,7,8\}}$, $q_0 \mid_{\{1,2,6,7,8\}}=$ the restriction of $q_0$ to $\{1,2,6,7,8\}$. Consider the matrix $W_1^{(1)}$ with rows and columns 1, 2, 6, 7, 8,

$$W_1^{(1)} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \cdots & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \cdots & \frac{1}{5} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{5} & \frac{1}{5} & \cdots & \frac{1}{5} \end{pmatrix}.$$

$W_1^{(1)}$ is a stable matrix, and, therefore, $W_1^{(1)} \in G_{\Delta_2^{(1)},\Delta_1^{(1)}}$, where $\Delta_1^{(1)} = (\{1\},\{2\},\{6\},\{7\},\{8\})$ and $\Delta_2^{(1)} = (\{1,2,6,7,8\})$ — a trivial case for the $G$ method, more exactly, for Theorem 1.10. Further, from

$$\left(q_1^{(1)}\right)' = W_1^{(1)} \left(q_0^{(1)}\right)' = W_1^{(1)} \left(q_0 \mid_{\{1,2,6,7,8\}}\right)',$$

we obtain

$$q_1^{(1)} = \left(\frac{1}{5} \sum_{i\in\{1,2,6,7,8\}} q_{0i}, \frac{1}{5} \sum_{i\in\{1,2,6,7,8\}} q_{0i}, ..., \frac{1}{5} \sum_{i\in\{1,2,6,7,8\}} q_{0i}\right),$$

$q_1^{(1)}$ is the value vector at time 1 for the subgraph $\mathcal{W}_5$. Second, we consider the subgraph of $\mathcal{G}$ with vertices 1, 3, 4, 5 and isomorphic to the complete graph

$\mathcal{K}_4$. For this subgraph, we consider that the initial value vector is

$$\left(\frac{1}{5}\sum_{i\in\{1,2,6,7,8\}} q_{0i}, q_{03}, q_{04}, q_{05}\right) := q_0^{(2)}.$$

Consider the matrix $W_1^{(2)}$ with rows and columns 1, 3, 4, 5,

$$W_1^{(2)} = \begin{pmatrix} \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}.$$

$W_1^{(2)}$ is also a stable matrix. Further, from

$$\left(q_1^{(2)}\right)' = W_1^{(2)}\left(q_0^{(2)}\right)',$$

we obtain

$$q_1^{(2)} = \left(\frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i},\ \frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i},\ \frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i},\ \frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i}\right),$$

$q_1^{(2)}$ is the value vector at time 1 for the above subgraph. Consider that the vertex 1 (it can be a leader) transmits the average $\frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i}$ to the vertices 2, 6, and 8 and, further, that the vertex 2 (this can also be a leader) or 6 transmits the above average to 7 — and thus distributed averaging is performed. Note that we can use $\mathcal{K}_5$ instead of the subgraph of $\mathcal{G}$ with vertices 1, 3, 4, 5 and isomorphic to $\mathcal{K}_4$...; finally, the vertices 1 and 2 (these can be leaders) will transmit the average $\frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i}$ to the vertices 6, 7, and 8. The study of this alternative way is left to the reader.

For the (nondirected simple finite) connected graphs with at least two vertices, distributed averaging based on the DeGroot model on distributed systems and the uniform (random) generation based on Markov chains of the vertices of a graph taking its edge set into account (when $\{i, j\}$ $(i \neq j)$ is not a edge of the graph, any transition matrix we use has the entries $(i, j)$ and $(j, i)$ equal to 0) are related problems/topics (...the stochastic matrices from Example 3.4 can be used to construct a Markov chain for the uniform generation of vertices of 3-cube graph...)

— *which of them is harder?...*

In this article, the framework for the DeGroot model and that for the DeGroot model on distributed systems we used are general, *i.e.*, homogeneous (when we use just one matrix at any time/step) and nonhomogeneous (when we use at least two different matrices at least two different times)

— *which of them are better?*

We believe that the nonhomogeneous frameworks are better — these frameworks are supported, among other things, by the DeGroot submodel, Theorem 3.3 (Example 3.4 is for $m = 3$), and Example 3.6 (we used nonhomogeneous products of matrices in these three places). For the homogeneous frameworks, see, *e.g.*, [7], [27], and [30].

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# ON CERTAIN RESULTS RELATED TO MARTINELLY-BOCHNER INTEGRAL

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**Abstract** In this note we provide some analogues of our numerous recent results on traces and distances in terms of Martinelly-Bochner integrals and kernels. These are first results of this type in terms of such kernels. Some assertions for Martinelly-Bochner integrals related with Holder classes and Lebegues points will be also discussed.

In recent years various new sharp results on traces and distances were provided in a big series of papers of the author. In all these papers properties of Bergman-type kernels and Bergman-type integral representations are playing a critical role. The intension of this paper to find some analogues of these results in terms of or with the help of more general integral representations and more general kernels in analytic function spaces in higher dimension so-called Martinelly-Bochner integral representations and Martinelly-Bochner kernels in $C^n$.

Our work consists of three parts. In the first part we partially generalize our results on traces. In the second part we provide estimates of distance function in terms of Martinelly-Bochner kernels and Martinelly-Bochner integrals. In the third part we present results on Martinelly-Bochner integrals related with Holder classes and Lebegues points. This type of issues arise naturally in view of recent series of papers and new results of the· author on multifunctional analytic spaces and related issues.

In our proofs we modify the methods of earlier results and theorems for the case of Martinelli-Bochner integrals and kernels. In particular we give full analogs of trace theorems in analytic Bergman-type spaces in strictly pseudoconvex domains with smooth boundary.

**Keywords:** Martinelly-Bochner integrals and kernels, analytic function, traces, distances, pseudoconvex domains.
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## 1. INTRODUCTION

In recent years various new sharp results on traces and distances were provided in a big series of papers of the· author (see [4], [9], [10], [11], [12], [13]). In all these papers properties of Bergman type kernels and Bergman-type integral representations are playing a critical role. The intension of this paper to find some analogues of these results in terms of or with the help of more general integral representations and more general kernels in analytic function spaces in higher dimension so-called Martinelly-Bochner integral representations and Martinelly-Bochner kernels in $C^n$ (see [5]). Some new results on Martinelly-Bochner integrals related with the Lebegues points and Holder functional classes will be also presented.

## 2. TRACES IN PSEUDOCONVEX DOMAINS AND MARTINELLY-BOCHNER INTEGRALS.

In this section we partially generalize our results on traces. The goal of this section is to provide a class of analytic functions on products of bounded strictly pseudoconvex domains $D\times\cdots\times D$ so that their traces allows Matrinelly-Bochner integral representation. To be more precise we first need standard definitions (see [5] and references there).

We define the unit ball in $C^n$ by $B$. Let $\tilde{U}$ be an open set in $\mathbb{C}^n$, we say $f \in C^k(\tilde{U})$ if $f$ is complex valued function and $f^{(k)} \in C^0(\tilde{U}) = C(U)$. We denote by $H(\tilde{U})$ the set of all analytic in $\tilde{U}$ functions. $\tilde{U}^m = \tilde{U} \times \cdots \times \tilde{U}$, $m \geq 1$ is a product domain and $H(\tilde{U}^m)$ is a space of all analytic function on $(\tilde{U}^m)$ (see [4], [9], [10], [11], [3]) or $D^m$. Using definitions of analytic function spaces in a domain, we define analytic function spaces in product domains in a natural way (see [10]).

We denote by $D$ in $\mathbb{C}^n$ a region with $\partial D$ boundary of $C^k$ class in some neighborhood of closure of $D$ and $d\rho \neq 0$ on $\partial D$. The $\rho$ function we call as usual the defining function of $D$ (see [10]). We consider an outer differential form (Bochner-Martinelly kernel) $U(\zeta, z)$ of $(n, n-1)$ type of the following form (see [5])

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} d\bar{\zeta}[K] \wedge d\zeta$$

where $d\bar{\zeta}[K] = d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{k-1} \wedge d\bar{\zeta}_{k+1} \wedge \cdots \wedge d\bar{\zeta}_m$, $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$. For $n = 1$ we have Causchy kernel $U(\zeta, z) = \frac{1}{2\pi i}\left(\frac{d\zeta}{\zeta - z}\right)$. Obviously coefficients of $U$ are harmonic in $C^n \backslash \{z\}$ and $d_\zeta U(\zeta, z)$=0.

Moreover (see [5]) if $g(\zeta, z)$ is a fundamental solution of Laplace equation then we have

$$U(\zeta, z) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial g}{\partial \zeta_k} d\bar{\zeta}[K] \wedge d\zeta = (-1)^{n-1} \partial_\zeta g \wedge \sum_{k=1}^{n} d\bar{\zeta}[K] \wedge d\zeta[K]$$

where $\partial = \sum_{k=1}^{n} \partial\zeta_k \frac{\partial}{\partial \zeta_k}$.

We denote various constants in estimates in this paper by C or by C with indexes, these constants are independent of functions in estimates.

**Theorem A.**(see [5]). *Let $D$ be a bounded region in $\mathbb{C}^n$ with partially-smooth boundary and let $f$ be holomorphic in $D$ of $C(\bar{D})$ class.Then*

$$\int_{\partial D} f(\zeta)U(\zeta,z) = \begin{cases} f(z), \ z \in D \\ 0, \ z \ z \notin \bar{D}. \end{cases} \tag{1}$$

This is a Martinelly-Bochner integral representation.

This theorem was obtained by Bochner then by Bochner then by Martinelly independently and by different methods. This formula is now classical and can be seen in various textbooks on complex analysis. Note for $n = 1$ we have classical Causchy formula, but for $n > 1$ we can easily see that the $U(\zeta, z)$ is not analytic by $z$ or $\zeta$. We will need the mentioned formula for Hardy spaces. We need some definitions. Below we omit the index of space if it is zero. Let $D$ be a bounded region, $\partial D \in C^{1+\alpha}$, $\alpha > 0$. We say that analytic function $f$ belongs to Hardy space $H^p(D)$, $p > 0$ if

$$\sup_{\varepsilon>0} \int_{\partial D} |f(\zeta - \varepsilon\nu(\zeta))|^p d\sigma(\zeta) < \infty,$$

where $d\sigma$ is the Lebesgue measure on $\partial D$ and $\nu(\zeta)$ is the vector field of outer normals to $\partial D$ (see [5]). For bounded pseudoconvex $D$ domains we have another definition, for a defining function $\rho$, $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$; let $D_\varepsilon = \{z \in D : \rho(z) < -\varepsilon\}$ for $\varepsilon > 0$. Then

$$H^p(D) = \{f \in H(D) : \sup_{\varepsilon>0} \int_{\partial D_\varepsilon} |f(\zeta)|^p d\sigma_\varepsilon < \infty\};$$

$0 < p \leq \infty$.

We define the same space in product domains in a natural way as we define Hardy spaces in polydisk using definitions in the unit disk see [3], [4].

**Proposition B.** (see [5]). *Let $D$ be bounded strongly pseudoconvex domain with smooth boundary. For all $p \geq 1$ and all $f$, $f \in H^p(D)$ Martinelly-Bochner representation is valid.*

*Proof.* Note first (see [5]) our proposition is valid for all bounded domains with $\partial D \in C^{1+\alpha}$, $\alpha > 0$. For each $f$, $f \in H^p$, $p \geq 1$, $f$ has boundary values almost everywhere on $\partial D$ so that these values are in $L^p(\partial D)$. By this classical fact, together with representation via boundary values

$$f(z) = \int_{\partial D} f(\zeta)P(\zeta, z)d\sigma,$$

where P is Poisson kernel of $D$ domain.

Note the $G$ Green function admits representation $G(\zeta, z) = g(\zeta, z) + h(\zeta, z)$, where $g$ is a solution of Laplace equation and where $h$ for all fixed $z \in D$ is a harmonic function in $D$ of $C^{1+\alpha}(\bar{D})$ class hence

$$P(\zeta, z)d\sigma = U(\zeta, z)/\partial D + \sum_{k=1}^{n}(-1)^{k-1}\frac{\partial h}{\partial\zeta_k d\bar{\zeta}}[K] \wedge d\zeta/\partial D = \Phi_1 + \Phi_2$$

but note

$$\int_{\partial D}(f(\zeta))\Phi_2(\zeta) = \int_{\partial D} f(\zeta)\sum_{k=1}^{n}(-1)^{k-1}\frac{\partial h}{\partial\zeta_k}d\bar{\zeta}[K] \wedge d\zeta$$

$$= \int_D f(\zeta)d\left(\sum_{k=1}^{n}(-1)^{k-1}\frac{\partial h}{d\zeta_k}d\bar{\zeta}[K] \wedge d\zeta\right) = 0.$$

This follows directly from the fact that $\sum_{k=1}^{n}(-1)^{k-1}\frac{\partial h}{d\zeta_k}d\bar{\zeta}[K] \wedge d\zeta$ is a closed form (see [5]). And this gives directly what we need for each $f$, $f \in H^p(D)$, $p \geq 1$ Martially-Bochner represention is valid. ∎

The intention of this section to show such type results for other spaces in unit ball or general bounded strictly pseudoconvex domains $D$, based on our previous results and on embeddings from [1].

The natural question is the following:

Let $X$ be a space of analytic functions, $X \subset H(D^m)$. Let also $Tr\ X = \{f(z, \dots, z), f \in X\}$. Can we say there is a function $g_0$, $g_0 \in Tr\ X$ so that it admits Martinelly-Bochner integral representation (1). Below we provide a general scheme of a solution for case of unit ball and bounded domains.

We define for a $\tilde{D}^\alpha$ - differential operator in pseudoconvex domain $D$ (see [1]), $0 < p, q < \infty$ Hardy-Sobolev and Bergman–Besov spaces of analytic functions

$$H^p_{\alpha,\beta}(D) = \left\{f \in H(D) : \sup_{\varepsilon>0}\left(\int_{\partial D_\varepsilon}|\tilde{D}^\alpha f(\zeta)|^p d\sigma_\zeta\right)^{\frac{1}{p}}\varepsilon^\beta < \infty\right\},\ \beta \geq 0,\ \alpha > 0.$$

$$A^{p,q}_{\delta,k}(D) = \left\{f \in H(D) : \sum_{|\alpha|\leq k}\int_0^{r_0}\left(\int_{\partial D_r}|\tilde{D}^\alpha f(\zeta)|^p d\sigma_\zeta\right)^{\frac{q}{p}} r^{\delta\frac{q}{p}-1}dr < \infty\right\},$$

$\delta > 0,\ \alpha > 0.$

We define same spaces in product domains in a natural way as we did in polydisk using definitions of spaces in the unit disk see [2], [3], [4].

We give a typical result in this direction on traces. Note for case of unit ball we have (see [2], [3], [4]).

**Theorem C.**

1 Let $p \leq 1$, $\beta \geq \alpha \geq 0$, then $A^{p,p}_{nm+(\beta-\alpha)pm-n}(B) \subset Trace\ H^p_{\alpha,\beta}(B^m)$;

2 Let $p \leq 1$; $\alpha_j \geq 0$, $t \geq \frac{\sum_{j=1}^m \alpha_j}{m}$ then

$$A^{p,p}_{mt+mn-n-\sum_{j=1}^m \alpha_j}(B) \subset Trace\ (A^{p,p}_{t,\vec{\beta}}(B^m));$$

$\beta_j = \alpha_j + 1$, $j = 1, \ldots, m$.

Obviously we have also (see [1], [3], [4]) the following embeddings ($0 < p \leq \infty$, $p \geq q$ and $s_1$, $s_2 \geq 0$).

$$A^p_{p(s_2-s_1),s_2}(B) \subset H^p_{s_1}(B) \subset H^p(B) \subset H^q(B)$$

then we have ($A^{p,p} = A^p$, $H^p_0 = H^p$)

$$A^p_{\alpha,\beta}(B) \subset A^p_{\alpha,0}(B) = A^p_\alpha(B),\ \beta \geq 0.$$

Now according to proposition B for any analytic $f$ function, $f \in H^p(B)$, $1 \leq p < \infty$ the Martinelly-Bochner integral representation is valid. This in combination with embeddings above gives

**Theorem 1.**

1 *Let $p \leq 1$, $\alpha \geq \beta$ ,then $Trace H^p_{\alpha,\beta}(B^m)$ contains a $f$ function for which*

$$f(z) = \int_{\partial D} f(\zeta)U(\zeta, z);\ z \in D \tag{M}$$

2 If $p \leq 1$, $\alpha_j \geq 0$, $\beta_j = \alpha_j + 1$, $t \geq \frac{\sum \alpha_j}{m}$, $j = 1, \ldots, m$, *then* $Trace\ A^p_{t,\vec{\beta}}(B^m)$ *contain a $f$ function for which (M) integral representation is valid.*

**Remark 1.** Note that previously this assertion was known for Bergman representation and Bergman kernels.

The case of general $D$ domains (pseudoconvex) can be considered similarly. It is based on our recent results from [11] and [12] and similar embeddings for pseudoconvex domains from [1]. Analytic Herz type spaces can be considered similarly (see [13]).

We need some definitions to formulate a theorem we need. Let $K(w)$ be entire function of one variable $w = u + iv$, $K : R \to R$, $K(u) \neq 0$ for all $u \in C$. Let

$$c_n K(Im\ z_n)\Phi(\zeta, z) = \frac{\partial^{n-2}}{\partial s^{n-2}} Im \frac{K(i\alpha + Im\ z_n)}{\alpha(i\alpha + Im\ (\zeta_n - z_n))},$$

$s = \sum_{k=1}^{n-1} |\zeta_k - z_k|^2$, $\alpha^2 = s + Re(\zeta_n - z_n)$, $c_n = (-1)^{n-1}(2\pi i)^n$.

**Theorem D1.**(see [5]). *If $n > 1$, then $\Phi(\zeta, z) = g(\zeta, z) + h(\zeta, z)$ ,where $\zeta \mapsto g(\zeta, z)$ is a fundamental solution of Laplace equation and $\zeta \mapsto h(\zeta, z)$ is harmonic in $C^n$, for each arbitrary fixed $z$.*

We need an extension of Martinelly-Bochner integral to unbounded domains $D$ in $C^n$ then for tubular domains the same problem can be posed and solved based on our recent results on traced in Bergman type spaces in tubular domains over symmetric cones with smooth boundary.

If $f \in H(D) \cap C(\bar{D})$ and if $h \in C'(\bar{D})$ and $h$ is harmonic by $\zeta$ for all $z \in D$. Then $\bar{\mu}_{\bar{h}}$ form is closed in $\bar{D}$

$$\bar{\mu}_{\bar{h}} = \sum_{k=1}^{m} (-1)^{n+k-1} \frac{\partial h}{\partial \zeta_k} d\bar{\zeta}[K] \wedge d\zeta$$

and

$$\int_{\partial D} f(\zeta)\bar{\mu}_{\bar{h}}(\zeta) = 0, \; f \in H(D) \cap C(\bar{D}).$$

Note in bounded domain then we have (see [5])

$$f(z) = \int_{\partial D} f(\zeta)[U(\zeta, z) + \bar{\mu}_{\bar{h}}(\zeta)], \; z \in D \qquad (G_1)$$

with coefficients $(K \times K)$ matrices $x, t \in R^m$.

**Remark 2.** Note if $K \equiv 1$ then $\Phi = g$.

**Theorem D2.**(see [5]). *If $D$ is unbounded domain in $C^n$, $n > 1$ with smooth boundary, $f \in H(D) \cap C(\bar{D})$ where $B(0, R)$ is a ball of radius $R$ then if*

$$\lim_{R\to\infty} \left( \int_{\partial D \setminus B(0,R)} f(\zeta)\Omega(\zeta, z) \right) = 0 \qquad (K_0)$$

*for all fixed $z \in D$ then*

$$f(z) = \int_{\partial D} f(\zeta)\Omega(\zeta, z) \qquad (K)$$

*where*

$$\Omega(\zeta, z) = \sum_{k=1}^{n} (-1)^{k-1} \left( \frac{\partial \Phi}{\partial \zeta_k} \right) d\zeta[\bar{k}] \wedge d\zeta.$$

*Proof.* Just use $G_1$ for domain $\tilde{D} \cap B(0, R)$ then pass $R \to \infty$. ∎

To formulate our theorem 2 we will need basic facts of theory of analytic function spaces in tubular domains over symmetric cones (see [20]).

A subset $\Omega$ of $\mathbb{R}^n$ or $V$, so that $dim\ V = n$ to be a cone if $\lambda x \in \Omega$, for all $x \in \Omega$, $\lambda > 0$, if $\lambda x + \mu y \in \Omega$ for all $x, y \in \Omega$, $\lambda, \mu > 0$ then it is convex. Let in addition $\Omega^* = \{y \in \mathbb{R}^n : (y/x) > 0, \forall x \in \bar{\Omega} \backslash \{0\}\}$ and $\Omega^* = \Omega$. This type open cone is selfdual ($\Omega^*$ is dual cone).

Let $G(\Omega) = \{g \in Gl(\mathbb{R}^n) : g\Omega = \Omega\}$, where $Gl(\mathbb{R}^n)$ denotes the group of all linear invertible transformation of $\mathbb{R}^n$. If for all $x, y \in \Omega$, $y = gx$, for some $g \in G(\Omega)$ then our open convex cone $\Omega$ is homogeneous, if also $\Omega^* = \Omega$ then it is symmetric cone.

If the equation $\Omega = \Omega_1 + \Omega_2$ is not possible for each $V_1 \subset \mathbb{R}^n$, $V_2 \subset \mathbb{R}^n$, then our cone is irreducible, here $V_i \neq \varnothing$, $i = 1, 2$ ($\Omega_1$, $\Omega_2$ are symmetric cones), where also $\Omega_i \subset V_i$, $i = 1, 2$.

We need also determinant $\Delta^t(Im\ z)$, $z \in \mathbb{C}^n$, $t \in (0, \infty)$. We fix $V$ - a simple Euclidean Jordan algebra with rank $r$.

If $x \in V$, $m(x) = \min\{k > 0 : (l, x, x^2, \dots, x^k)$ are linearly dependent$\}$, then $1 \leq m(x) \leq dim\ V$ and $r = \max\{m(x) : x \in V\}$, we say rank of $V$ is $r$.

According to spectral theorem if $V$ has rank $r$, then $x = \sum_{i=1}^r \lambda_i c_i$; $\lambda_i \in R$; $c_i$ - are elements of so called Jourdan frame, and $\{\lambda_i\}$ are determined uniquely by $x$ (with their multiplicities). We fix now a Peirce decomposition of $V = \oplus_{1 \leq i \leq j \leq r} V_{ij}$; (we formally look at V as a space of symmetric matrices $(V_{ij})$, $V_{ii} = Rc_i$, where $R$ is a special mapping, $V_{ij} = V(c_i, 1/2) \cap V(c_i, 1/2) = \{x \in V : c_i x = c_j x = \frac{x}{2}\}$, $i < j$, $dim\ V_{ij} = d = 2\frac{n/r-1}{r-1}$). We denote by $P_{ij}$ the orthogonal projection of $V$ onto $V_{ij}$ for $i \leq j$. Finally we denote by $\Delta_j(x)$, $j = 1, \dots, r$, the principal minors of $x \in V$ with respect to the fixed Jordan frame $\{c_1, \dots, c_r\}$. That is $\Delta_k(x)$ is the determinant of the projection $P_k x$ of $x$ in the Jordan subalgebra $V^{(k)} = \oplus_{1 \leq i \leq j \leq r} V_{ij}$.

It is well-known that $\Omega = \{x \in V : \Delta_k(x) > 0 : k = 1, \dots, r\}$. We have also $\Delta_k(mx) = \Delta_k(x)$, $x \in V$, $m \in Z_+$, $m > 0$. See other properties of $\Delta_k$ in [20].

We define $\Delta_s(x) = \prod_{j=1}^r \Delta_j^{s_j - s_j + 1}(x) = \Delta_1^{s_1 - s_2}(x) \dots \Delta_r^{S_r}(x)$, $x \in \Omega$, $s \in C^r$.

We have that $|\Delta_s| = \Delta(Im\ z)$ and $\Delta_s \sum_{i=1}^r a_i c_i = \prod_{i=1}^r a_i^{s_i}$; $a_i > 0$, $i = 1, \dots, r$.

Let $\Omega$ be an irreducible symmetric cone in the Euclidean space $V$, and $T_\Omega = V + i\Omega$ the corresponding tube domain in the complexified space $V^{\mathbb{C}}$, $T_\Omega^m = T_\Omega \times \dots T_\Omega$.

As combination of this theorem D2 and theorems on traces [16] in tube we have

**Theorem 2.** *Let $T_\Omega$ be a tubular domain over symmetric cone, and let*

$$A^p_\alpha(T^m_\Omega) = \left\{ f \in H(T^m_\Omega) : \int_{T_\Omega} \cdots \int_{T_\Omega} |f(z_1, \ldots, z_m)|^p \prod_{j=1}^m \Delta^\alpha(Im\ z_j) dv(z_j) < \infty \right\},$$

$0 < p < \infty$, $\alpha > -1$. *Then* $Trace A^p_\alpha = \{f(z, \ldots, z),\ f \in A^p_\alpha(T^m_\Omega)\}$ *contains a function, so that representation (K) is valid if ($K_0$) holds.*

*Proof.* Sharp traces of Bergman spaces in products of tubular domains were calculated by the author, in [16], this set contains analytic $f$ functions from a Bergman space in tubular domain. This with combination with the mentioned result gives us the proof immediately. ∎

Similar results can be obtained based on integral representation based on second Green formula (see [5]) and our recent results on traces of Bergman harmonic function spaces in the unit ball (see [17]) $B = \{|x| < 1\} \subset R^n$ and $R^{n+1}_+$.

We give more details on this integral representation.

Let in $R^m$, $m \geq 1$ us consider a differential operator in $R^m$ $A = \sum_{j=1}^m A_j \frac{\partial}{\partial x_j}$, where $A_j$ are matrices of order $l \times K$ on $C$, $S_m = \{|x| = 1, x \in R^m\}$ if

$$A_i^* A_j + A_j^* A_i = \begin{cases} 0 & \text{if } i \neq j \\ 2I_k & \text{if } i = j. \end{cases}$$

where $i, j \in [1, n]$, $A_i^* = \bar{A}_i^T$ complex conjugate of $A_i$, $I_k$ is unit matrix. Then $A$ is Dirac operator. For solutions of Dirac operator we have analogue of Martinelly-Bochner theorem

For definition of vector function space in theorem E we refer to [5].

**Theorem E.**(see [5]).

*Let $A$ be Dirac operator, $D \subset R^n$ is a bounded domain with smooth boundary. If $f \in [C(\bar{D}) \cap C^1(D)]^k$, $Af = 0$, then*

$$\int_{\partial D} U_A(t, x) f(t) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{if } x \notin \bar{D}. \end{cases}$$

*where $U_A$ is a special Martinelly-Bochner type kernel in $R^m$ see [5] a differential form type of $(n-1)$ type*

$$U_A(t, x) = \frac{1}{Vol(S_m)} \sum_{j=1}^m \sum_{i=1}^m (-1)^{i-1} A_j^* A_i \frac{t_j - x_j}{|t - x|^m} dt[i].$$

Details of proofs of these assertions analogues of previous theorems 1,2 for harmonic function spaces based on theorem E we leave to readers (see [12], [13]).

## 3. DISTANCE FUNCTIONS AND MARTINELLY-BOCHNER INTEGRALS

The goal of this section to provide estimates of distance function in terms of Martinelly-Bochner kernels and Martinelly-Bochner integrals based on methods and results of [9], [10] and [11] and our earlier work cited there.

Note that previously we obtained such type theorems in various spaces of analytic and harmonic functions of one and several variables in various domains in terms of Bergman kernel and Bergman-type projections (see [9], [10], [11] and various references there). The line of our proof is rather similar however to those simpler cases but it is based on some results on more general kernels as Martinelly-Bochner kernels and Martinelly-Bochner integrals (see [5] and references there).

Let $D$ be bounded domain in $C^n$. $X$, $Y$ be quazinormed subspaces of $H(D)$, where $H(D)$ is the space of all analytic functions in $D$. Let $X \subset Y$, let $f \in Y$. We search for those estimates of a function $dist_Y(f, X)$ which generalize previously known such type estimates obtained via similar Bergman type kernels.

We have the following result in this direction. First if $X \subset H^p$, $p \geq 1$ then the problem of estimates of $dist_{H^p}(f, X)$, $f \in H^p$ appears naturally. In our previous papers to role of Bergman type kernels for this problem was critical.

Let $H^p \subset X$, $X \subset H(D^m)$, then let $f \in X$. We also can consider a problem of finding estimates of function $dist_X(f, H^p)$, $1 \leq p < \infty$, $H^p = H^p(D^m)$ on product domains in terms of Martinelly-Bochner kernels or integrals.

From the arguments used in the proofs of our previous papers [10], [11], [14], [15], we may now give a useful distance evaluation result.

Let $X \subset H^p(D^m)$, $1 \leq p < \infty$, $m \in N$, we assume $X$ is a quazinormed subspace of $H(D^m)$. To estimate $dist_{H^p}(f, X)$, $f \in H^p$, $1 \leq p < \infty$ we follow our arguments from [14] in the unit disk $\{|z| < 1\}$ and unit ball.

Since $f \in H^p(D^m)$, then applying Martinelly-Bochner representation by each variable

$$f(z_1, \dots, z_m) = \int_{(\partial D)^m} f(\vec{\zeta})U(\vec{\zeta}, \vec{z}) = \int_{(\partial D)} \cdots \int_{\partial D} f(\zeta_1, \dots, \zeta_m) \prod_{j=1}^{m} U(\zeta_j, z_j);$$

$z_j \in D$, $j = 1, \dots, m$ where

$$H^p(D^m) = \{f \in H(D^m) : \sup_{R>0} \int_{\partial D} \cdots \int_{\partial D} |f(\vec{\zeta})|^p d\sigma < \infty\},\ 1 \leq p < \infty$$

is a Hardy space on products of $D$ domains, $m \in N$, $D$ is bounded in $C^n$, $\partial D \in C^{1+\alpha}$, $\alpha > 0$ (note that these are Banach spaces) and where $\vec{\zeta}_R = (\zeta_1 - R\nu(\zeta_1), \dots, \zeta_m - R\nu(\zeta_m))$, $\nu$ is a vector field of outer unit normals to $\partial D$.

Let $X^{\Psi}_{\varepsilon,f} = \{\zeta \in \partial(D^m) : \Psi(\zeta)|f(\zeta)| \geq \varepsilon\}$; $\Psi \in C^1(\partial(D^m))$ fixed. Then $f = f_1 + f_2$

$$f_1(z) = \int_{X^{\Psi}_{\varepsilon,f}} f(\zeta) \prod_{j=1}^{m} U(\zeta_j, z); \qquad f_2(z) = \int_{\partial(D^m)\setminus X^{\Psi}_{\varepsilon,f}} f(\zeta) \prod_{j=1}^{m} U(\zeta_j, z).$$

Assuming $f_2 \in H^p$, we get

$$dist_{H^p}(f, x) \leq c \inf \left\{ \varepsilon > 0 : \| \int_{X^{\Psi}_{\varepsilon,f}} f(\zeta) \prod_{j=1}^{m} U(\zeta_j, z) \|_{X(D^m)} < \infty \right\},$$

since $dist_{H^p}(f, x) \leq c\|f - f_1\|_{H^p} = \|f_2\|_{H^p}$.

**Theorem 3.** *Let $p \geq 1$. Let $\Psi \in C^1(\partial(D^m))$, $\Psi \geq 0$, be fixed function, let $f \in H^p(D^m)$, $\int\limits_{\partial(D^m)\setminus X^{\Psi}_{\varepsilon,f}} f(\zeta) \prod_{j=1}^{m} U(\zeta_j, z) \in H^p$. Then*

$$dist_{H^p}(f, Y) \leq c \inf \left\{ \varepsilon > 0 : \left\| \int_{X^{\Psi}_{\varepsilon,f}} f(\vec{\zeta}) \prod_{j=1}^{m} U(\zeta_j, z) \right\|_{Y(D^m)} < \infty \right\}$$

*where*

$$X^{\Psi}_{\varepsilon,f} = \{\zeta \in \partial(D^m) : \Psi(\zeta)|f(\zeta)| \geq \varepsilon\},$$

*where $Y \subset H^p$ is a quazinormed subspace of $H^p(D^m)$ for any bounded $D$ domain with $\partial D \in C^{\alpha+1}$, $\alpha > 0$ boundary.*

**Remark 3.** Similar results are valid for Martinelly-Bochner type integrals and harmonic function spaces in $R^{n+1}_+$ and unit ball in $R^m$. See [17] where similar problems where solved via Bergman kernel.

Theorem 3 is a typical assertion, other assertions of such type in terms of $U(z, \zeta)$ kernels can be also formulated similarly.

## 4. SOME REMARKS ON MARTINELLY-BOCHNER INTEGRALS RELATED TO HOLDER CLASSES AND LEBESGUE POINTS.

The goal of this section is among other things from [5] related with expressions like $|\Phi_f(z_1) - \Phi_f(z_2)|$ to the case of two functional similar type expressions like $|\Phi_f(z_1)-\Phi_g(z_2)|$, where $\Phi$ is a certain operator, $f$, $g$ are certain functional class members, $z_1$, $z_2 \in D$, where $D$ is a fixed domain in $C^n$.

This type of issues arise naturally in view of recent series of papers and new results of the author on multifunctional analytic spaces and related issues (see [8], [17] and various references there).

To obtain this natural tranfsormation of known results from one functional case to two functional case we follow the proof of [5] and at the same time modify proofs from there to our case. First we provide certain equalities (A), (B), (C), (D), which can be checked directly, then use them in our proofs. We alert the reader that in those places where arguments are close or the same as in [5], we omit details making the exposition of this note rather concise we note also that our problems we considered below is an attempt to provide natural extensions of known one functional results to two functional case.

Let

$$(Mf)(z^+) = \int_{\partial D} f(\zeta)U(\zeta, z^+);$$

$$(Mf)(z^-) = \int_{\partial D} f(\zeta)U(\zeta, z^-);$$

then we have following equalities

$$(Mf)(z^+)-(Mg)(z^-)-f(z)-g(z) = \int_{\partial D}(f(\zeta)-f(z))(U(\zeta,z^+)-U(\zeta,z^-))-$$

$$-\int_{\partial D} g(\zeta)U(\zeta,z^-)-\int_{\partial D} g(z)(U(\zeta,z^+)-U(\zeta,z^-))+\int_{\partial D} f(\zeta)U(\zeta,z^-) = \quad (A)$$

$$= \int_{\partial D}(f(\zeta)-f(z))(U(\zeta,z^+)-U(\zeta,z^-))+M(f-g)(z^-)-g(z)$$

$$(Mf)(\tilde z^+)-(Mg)(\tilde z^-) = \int_{\partial D}(f(\zeta)-f(z^0))U(\zeta,\tilde z^+)-$$

$$-\int_{\partial D}(f(\zeta)-f(z^0))U(\zeta,\tilde z^-)+f(z^0)\int_{\partial D}(U(\zeta,\tilde z^+)-U(\zeta,\tilde z^-))- \qquad (B)$$

$$-\int_{\partial D} g(\zeta)U(\zeta,\tilde z^-)+\int_{\partial D} f(\zeta)U(\zeta,\tilde z^-) = \int_{\partial D}(f(\zeta)-f(z^0))U(\zeta,\tilde z^+)-$$

$$-\int_{\partial D}(f(\zeta)-f(z^0))U(\zeta,\tilde z^-)+f(z^0)\int_{\partial D}(U(\zeta,\tilde z^+)-U(\zeta,\tilde z^-))+M(f-g)(\tilde z^-);$$

we omit technical calculations leaving them to readers. Note all equalities are based on simple properties of Martinelly-Bochner integrals (see [5]).

It can be checked directly that

$$\int_{\partial D}(f(\zeta)-f(0))U(\zeta,z)-\int_{\partial D\setminus B(0,\varepsilon)}(g(\zeta)-g(0))U(\zeta,0) =$$

$$= \int_{\partial D\setminus B(0,\varepsilon)} (f(\zeta)-f(0))(U(\zeta,z)-U(\zeta,0)) + \int_{\partial D\cap B(0,\varepsilon)} (f(\zeta)-f(0))U(\zeta,z) -$$

$$- \int_{\partial D\setminus B(0,\varepsilon)} (g(\zeta)-g(0))U(\zeta,0) + \int_{\partial D\setminus B(0,\varepsilon)} (f(\zeta)-f(0))U(\zeta,0) =$$

$$= \int_{\partial D\setminus B(0,\varepsilon)} (f(\zeta)-f(0))(U(\zeta,z)-U(\zeta,0)) + \tag{C}$$

$$+ \int_{\partial D\cap B(0,\varepsilon)} (f(\zeta)-f(0))U(\zeta,z) + \int_{\partial D\setminus B(0,\varepsilon)} ((f-g)(\zeta)-(f-g)(0))U(\zeta,0);$$

$$- \int_{\partial D\setminus\sigma_\delta} (f(\zeta)-f(z^1))U(\zeta,z^1) + \int_{\partial D\setminus\sigma_\delta} (f(\zeta)-g(z^2))U(\zeta,z^2) =$$

$$= \int_{\partial D\setminus\sigma_\delta} (g(\zeta)-g(z^2))(U(\zeta,z^2)-U(\zeta,z^1)) + (f(z^1)-f(z^2)) \int_{\partial D\setminus\sigma_\delta} U(\zeta,z^1) + \tag{D}$$

$$+ \int_{\partial D\setminus\sigma_\delta} (g(\zeta)-g(z^2))U(\zeta,z^1) - \int_{\partial D\setminus\sigma_\delta} f(\zeta)U(\zeta,z^1) + f(z^2) \int_{\partial D\setminus\sigma_\delta} U(\zeta,z^1) =$$

$$= \int_{\partial D\setminus\sigma_\delta} (g(\zeta)-g(z^2))(U(\zeta,z^2)-U(\zeta,z^1)) +$$

$$+(f(z^1)-f(z^2)) \int_{\partial D\setminus\sigma_\delta} U(\zeta,z^1) + A_{f,g} + B_{f,g} + C_{f,g} =$$

$$= \int_{\partial D\setminus\sigma_\delta} (g(\zeta)-g(z^2))(U(\zeta,z^2)-U(\zeta,z^1)) + (f(z^1)-f(z^2)) \int_{\partial D\setminus\sigma_\delta} U(\zeta,z^1) +$$

$$+ \int_{\partial D\setminus\sigma_\delta} ((g-f)(\zeta)-(g-f)(z^2))U(\zeta,z^1);$$

where $\sigma_\delta$ is arc on $\partial D$.

**Remark 4.** For those cases when $g = f$ all equalities (A), (B), (C), (D) can be seen in [5] (see lemma 5.3, theorem 6.1, 6.2).

Let $\Omega$ be a domain, we say $f$ is Holder class $\alpha$ if $|f(\zeta)-f(z)| \le c|\zeta-z|^\alpha$, $\zeta, z \in \Omega$, $\alpha > 0$. We consider bounded domains with smooth boundary, $f \in L^1(\partial D)$, $D = \{z \in C^n, \rho(z) < 0\}$, $\rho \in C^1(C^n)$, $d\rho \ne 0$ on $\partial D$. Let $V(\partial D)$ be a neighborhood of $\partial D$. We assume $f$ can be extended to $V(\partial D)$. Consider integrals

$$\Phi(z) = \int_{\partial D} (f(\zeta)-f(z))U(\zeta,z).$$

Note if $z \notin \partial D$ then the integral has no singularity, if $z \in \partial D$, then

$$|f(\zeta) - f(z)| \times |U(\zeta, z)| \le c|\zeta - z|^{\alpha+1-2n} d\sigma(\zeta) \tag{2}$$

if $f$ is $\alpha$ Holder class function. The natural question is let $f$ is of Holder class $\alpha$ in $V(\partial D)$ then can we say $\Phi(z)$ is of same class $\alpha$. The answer is positive (see [5]).

The next more general question is if $f$, $g$ are of Holder class $\alpha$ then what can we say on $|\Phi_f(z_1) - \Phi_g(z_2)|$, $z_1, z_2 \in V(\partial D)$. We will use formula (D) for our proof. But first note that the following simple observation is valid.

Let $z^1$, $z^2 \in V(\partial D)$, $|z^1 - z^2| = \delta$, $\delta$ is small, if $B(z^1, 2\delta) \subset V(\partial D)$, $\sigma_\delta = \partial D \cap B(z^1, 2\delta)$. Then (see [5])

$$|\int_{\sigma_\delta} (f(\zeta) - f(z^j) U(\zeta, z^j)| \le c_1 \int_{\sigma_\delta} |\zeta - z^j|^{1+\alpha-2n} d\sigma \le c_2 \delta^\alpha, j = 1, 2.$$

This follows from (2) directly and the fact that $\sigma_\delta$ is smooth.

So for $f = g$ case it is remains to estimate integrals by $D \backslash \sigma_\delta$ to get estimate for $|\Phi_f(z_1) - \Phi_f(z_2)|$. But for $|\Phi_f(z_1) - \Phi_g(z_2)|$ we have formula (D) and estimates

$$A_f = |\int_{\sigma_\delta} (f(\zeta) - f(z^1)) U(\zeta, z^1)| \le \tilde{c}_1 \delta^\alpha, \qquad 0 < \alpha < 1$$

$$B_g = |\int_{\sigma_\delta} (g(\zeta) - g(z^2) U(\zeta, z^2)| \le \tilde{\tilde{c}}_1 \delta^\alpha, \qquad 0 < \alpha < 1.$$

Since obviously we have

$$|\Phi_f(z_1) - \Phi_g(z_2)| = \left| \int_{\partial D} (f(\zeta) - f(z_1)) U(\zeta, z_1) - \int_{\partial D} (g(\zeta) - g(z_2)) U(\zeta, z_2) \right| \le$$
$$\le A_f + B_g + C_{f,g};$$

$$C_{f,g} = \left| \int_{\partial D \backslash \sigma_\delta} (f(\zeta) - f(z_1)) U(\zeta, z_1) - \int_{\partial D \backslash \sigma_\delta} (g(\zeta) - g(z_2)) U(\zeta, z_2) \right|,$$

using (D) we have now that $c_{f,g} \le c_g^1 + c_f^2 + c_g^3 + c_f^4 + c_f^5$. Note that

$$c_g^3 + c_f^4 + c_f^5 = \int_{\partial D \backslash \sigma_\delta} ((g - f)(\zeta) - (g - f)(z^2);\ U(\zeta, z^1) = \Phi_{f,g}(z_1, z_2).$$

**Remark 5.** Note we also have

$$|\Phi_{f,g}(z_1, z_2)| \le |\Phi_{g-f}(z_2)| + \tilde{c}^1_{g-f},$$

$$\tilde{c}^1_{g-f} = \int_{\partial D\setminus\sigma_\delta} ((g-f)(\zeta) - (g-f)(z^2))(U(\zeta,z^1) - U(\zeta,z^2)).$$

So we have that

$$|\Phi_f(z_1) - \Phi_g(z_2)| \le c^1_g + \tilde{c}^1_{g-f} + c^2_f + c\delta^\alpha + \tilde{c}|\Phi_{g-f}(z_2)|.$$

We will insist on estimate with $\Phi_{g,f}$.

We have $c^2_f \le c\delta^\alpha$ since $|\int_{\partial D\setminus\sigma_\delta}(U(\zeta,z^1))| \le 1$, see [5]. We estimate $c^1_g$ and $\tilde{c}^1_{g-f}$. This is based on simple estimates of $U(\zeta,z^2) - U(\zeta,z^1)$, $z_1, z_2 \in V(\partial D)$.

We have that (see [5])

$$\left|\int_{\partial D\setminus\sigma_\delta}(g(\zeta) - g(z^2))(U(\zeta,z^2) - U(\zeta,z_1))\right| \le$$

$$\le \tilde{c}\delta \int_{\partial D\setminus\sigma_\delta} |\zeta - z^1|^{\alpha-2n} d\sigma \le \tilde{c}(\delta^\alpha)$$

$$\left|\int_{\partial D\setminus\sigma_\delta}((f-g)(\zeta) - (g-f)(z^2))(U(\zeta,z^1) - U(\zeta,z^2))\right| \le$$

$$\le \tilde{\tilde{c}}\delta \int_{\partial D\setminus\sigma_\delta} |\zeta - z^1|^{\alpha-2n} d\sigma \le \tilde{\tilde{c}}(\delta^\alpha)$$

Since $\int_{\partial D\setminus\sigma_\delta} |\zeta - z^1|^{\alpha-2n} d\sigma \le \bar{c}\delta^{\alpha-1}$ (see [5]).

So finally we have an estimate

$$|\Phi_f(z_1) - \Phi_g(z_2)| \le c\delta^\alpha + |\Phi_{g,f}(z_1,z_2)|,\ 0<\alpha<1$$

and

$$|\Phi_f(z_1) - \Phi_g(z_2)| - |\Phi_{g,f}(z_1,z_2)| \le c\delta^\alpha.$$

Note if $f \equiv g$ this obtained in [5] and [6].

We now formulate the final result.

**Theorem 4.** *Let $f$, $g$ be of Holder class $\alpha$, $0<\alpha<1$ in $V(\partial D)$. Then in $V(\partial D)$ we have the following estimate*

$$|\Phi_f(z_1) - \Phi_g(z_2)| \le c\delta^\alpha + |\Phi_{g,f}(z_1,z_2)|;\ z_1,z_2 \in V(\partial D),$$

*$|z^1 - z^2| = \delta$, $\delta < \delta_0$, for some fixed $\delta_0$, $\delta_0 > 0$.*

We assume $Mf(z) = \int_{\partial D} f(\zeta)U(\zeta,z)$, $z \notin \partial D$, $f \in C^1(\bar{D})$ below.

Now we use (A), (B), (C) to obtain such type results for other mentioned assertions from [5]. Note to get the proof we must follow one functional proof from [5] modify it to use mentioned equalities (A), (B), (C).

Let $D$ be bounded domain with $C^1$ boundary and $f \in L^1(\partial D)$ and $z^0 \in \partial D$ and $V_{z^0}$ be a cone with $z^0$ as peak (vertex), with conic angle equal or less to $\pi/2$. Let $z \in D \cap V_{z^0}$. We say $z^0$ is a Lebegues point for $f$ if (see [5])

$$\lim_{\varepsilon \to +0} \varepsilon^{1-2n} \int_{\partial D \cap B(z^0,\varepsilon)} |f(\zeta) - f(z^0)| d\sigma = 0.$$

We will be using (C) and arguments from proof of theorem 5.2 [5]. We note first (see [5])

$$\left| \int_{\partial D \cap B(0,\varepsilon)} |f(\zeta) - f(0)| U(\zeta, z) \right| \leq K \int_{\partial D \cap B(0,\varepsilon)} |f(\zeta) - f(0)| d\sigma \to 0;$$

$K = \frac{c_2}{\varepsilon^{2n-1}}$ *as* $\varepsilon \to 0$.
This and (C) lead to theorem.

So based on (C) we have the following.

**Theorem 5.** *Let* $z \in D \cap V_{z^0}$ *be a Lebegues point for* $f$ *and* $g$, $f, g \in L^1(\partial D)$. *Then*

$$\lim_{\substack{z \to z^0 \\ z \in V_{z^0}}} \int_{\partial D} (f(\zeta) - f(z^0)) U(\zeta, z) - \int_{\partial D \setminus B(z^0, |z - z^0|)} (g(\zeta) - g(z^0) U(\zeta, z^0)) =$$

$$= \lim_{\substack{z \to z^0 \\ z \in V_{z^0}}} \int_{\partial D \setminus B(z^0, |z - z^0|)} ((f - g)(\zeta) - (f - g)(z^0)) U(\zeta, z^0).$$

Using (A) and (B) and following the proof of theorem 6.1, 6.2 (see [5]). We can have assertions concerning

$$\lim_{z \pm \to z^0} Mf(z^+) - Mg(z^-) - f(z^0) - g(z^0),$$

where, $f, g \in L^1(\partial D)$, $z^0$ is a Lebegues point of $f$ and $g$ simultaneously and

$$\int_{\partial D} |Mf(\tilde{z}^+) - Mg(\tilde{z}^-) - f(z) - g(z)|^p d\sigma$$

if $D$ is bounded, $\partial D \in C^1$, $f \in L^p(\partial D)$, $p \geq 1$, $\nu(\zeta)$ is a unit vector of outer normal to $\partial D$ on $\zeta$, $\tilde{z}^+ = z - \varepsilon \nu(z)$, $\tilde{z}^- = z + \varepsilon \nu(z)$. We omit details here.

So, based on (B), we have the following

**Theorem 6.** *Let* $z^0$ *be a Lebesgue point of* $f$ *and* $g$, $f \in L^1(\partial D)$, $g \in L^1(\partial D)$, $z^+ \in V_{z_0} \cap D$, $z^- = V_{z_0} \cap (C^n \setminus \bar{D})$, $a|z^+ - z^0| \leq |z^- - z^0| \leq b|z^+ - z^0|$,

*$a, b \in R$, $a \leq b$, $z_0 \in \partial D$, $D$ is bounded, $\partial D \in C^1$, $z^0$ is a peak of cone $V_{z_0}$, cone angle is $\beta < \pi/2$. Then*

$$\lim_{z\pm \to z^0}(Mf(z^+) - Mg(z^-)) = f(z_0) + \lim_{z\pm \to z^0}(M(f-g)(z^-)).$$

**Remark 6.** For $f = g$ this is a theorem from [5].

Based on (A) we have the following theorem 7.

**Theorem 7.** *Let $D$ be a bounded domain with smooth boundary. Let $f, g \in L^p(\partial D)$, $p \geq 1$. Let $\tilde{z}^+$, $\tilde{z}^-$ belong to $D$ and $C^n \backslash \bar{D}$ for $\varepsilon$ small enough. Then*

$$A_{f,g} = \int_{\partial D} |Mf(z - \varepsilon\nu(z)) - Mg(z + \varepsilon\nu(z))|^p d\sigma \leq$$

$$\leq c \int_{\partial D} |f|^p d\sigma + \int_{\partial D} |M(f-g)(z^-)|^p d\sigma;$$

$$\lim_{\varepsilon \to +0} \int_{\partial D} |Mf(z - \varepsilon\nu(z)) - Mg(z + \varepsilon\nu(z)) - f(z) - g(z)|^p d\sigma \leq$$

$$\leq \lim_{\varepsilon \to 0} \|M(f-g)(z^-)\|_{L^p} + \|g\|_{L^p}.$$

**Remark 7.** At the end of this paper we find interesting to suggest another idea related from one hand to Martinelly-Bochner integrals from the other hand to our recent work on traces in analytic functions Bergman spaces. In [7] the role of expanded Bergman projection based on expaned Bergman kernel was crucial for solution of certain problems related with traces of analytic functions in polyballs. It will be nice to study their analogues expanded Martinelly-Bochner kernels and based on them expanded Martinelly-Bochner integrals based in particular on the following simple idea.

Let as usual $g(\zeta, z) = c_n \frac{1}{|\zeta - z|^{2n-2}}$, $n > 1$, $\rho$ be defining function of $D \subset C^n$, $D = \{\rho < 0\}$, $d\rho \neq 0$ on $\partial D$, $\rho \in C^1$.

$P_{k,s}$ be a complete orthonormal system of homogeneous harmonic polynomials in $L^2(S)$, * is well-known Hodze operator for differential forms see [5] $\sigma(\zeta) = \sum\limits_{k=1}^{n} (-1)^{k-1} \bar{\zeta}_k d\bar{\zeta}[k] \wedge d\zeta$.

We suggest to study kernels of type (expanded Martinelly-Bochner kernels)

$$U(\zeta, z_1, \dots, z_m) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial}{\partial \zeta_k} \prod_{j>1}^{m} g(\zeta, z_j) d\bar{\zeta}[K] \wedge d\zeta$$

$$\tilde{U}(\zeta, z_1, \dots, z_m) = \frac{(n-1)!}{\pi^n} \sum_{k=1}^{n} \frac{\partial \rho}{\partial \zeta}_k \prod_{j=1}^{m} \frac{(\bar{\zeta}_k - \bar{z}_j^k)}{|\zeta - z_j|} d\sigma$$

$$\bar{U}(\zeta, z_1, \ldots, z_m) = \left( -\sum_{k,s} \frac{P_{k,s}(z_1)}{n+k-1} \left[ *\partial \frac{\overline{P_{k,s}(\zeta)}}{|\zeta|^{2n+2k-2}} \right] \right)$$

$$\cdots \left( -\sum_{k,s} \frac{P_{k,s}(z_m)}{n+k-1} \left[ *\partial \frac{\overline{P_{k,s}(\zeta)}}{|\zeta|^{2n+2k-2}} \right] \right).$$

Note that for $m = 1$ these kernels coincide with classical Martinenelly-Bochner integrals (see [5]). Same idea used for Bergman kernel in series of papers of the author (see [7] and references there).

$$B(z_1, \ldots, z_m, w) = \prod_{j=1}^{m} \frac{(1-|z_j|)^{\alpha_j}}{(1-\bar{z}_j w)^{\alpha_j+2}}, \ \alpha_j > -1, z_j, w \in D, \ j = 1, \ldots, m.$$

These types of extensions of Bergman kernel was considered by author in [7].

## 5. CONCLUSION.

In our paper we stated some analogues of our recent results on traces and distances in terms of Martinelly-Bochner integrals and kernels. These are first results of this type in terms of such kernels. We also discussed some assertions for Martinelly-Bochner integrals related with Holder classes and Lebegues points.

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